

# Growth to value: Option exercise and the cross section of equity returns <sup>☆</sup>

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## Abstract

We propose a general equilibrium model to study the link between the cross section of expected returns and book-to-market characteristics. We model two primitive assets: value assets and growth assets that are options on assets in place. The cost of option exercise, which is endogenously determined in equilibrium, is highly procyclical and acts as a hedge against risks in assets in place. Consequently, growth options are less risky than value assets, and the model features a value premium. Our model incorporates long-run risks in aggregate consumption and replicates the empirical failure of the conditional capital asset pricing model (CAPM) prediction. The model also quantitatively accounts for the pattern in mean returns on book-to-market sorted portfolios, the magnitude of the CAPM-alphas, and other stylized features of the cross-sectional data.

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## 1. Introduction

Historically, stocks with a high book-to-market ratio earn higher average returns than do those with a low book-to-market ratio. The difference in mean returns on value and growth stocks, the

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so-called value premium, is about 6% per year, which is known to pose a serious challenge to the standard asset pricing models such as the capital asset pricing model (CAPM). In this paper we offer a rational explanation of why growth stocks, which effectively are options on assets in place, are less risky and, therefore, carry a low premium relative to value stocks. We propose a general equilibrium model that accounts for the dispersion in average returns on value and growth stocks and for the failure of the CAPM predictions in the data.

We model growth assets as options on assets in place. Exercising an option requires one unit of capital goods and results in the creation of a new asset in place. Thus, growth options are long positions in assets in place and short positions in capital goods. If the cost of exercising an option is sensitive to macroeconomic risks, it acts as a hedge against risks in assets in place, making growth options less risky than value assets. We show that, in equilibrium, the marginal cost of capital goods is highly procyclical if their supply is scarce relative to the existing options and aggregate risk is mean reverting. The intuition behind our result can be explained as follows. If the economy is currently above its long-run trend, then it likely slows down in the future due to mean reversion, which makes delaying option exercise less attractive. Owners of growth assets who are trying to expedite the exercising of an option in good times drive up the price of capital goods. By the same logic, the cost of exercising an option is lower when macroeconomic conditions are unfavorable. The procyclical dynamics of the equilibrium price of capital goods partially offset the cyclical fluctuations in assets in place, which makes growth assets less vulnerable to aggregate risks and, thus, results in a value premium.

We embed the above mechanism in a long-run risk economy (Bansal and Yaron, 2004) and provide an endogenous link between dividend exposure to persistent risks in consumption at the aggregate level and differential exposure to long-run risks across book-to-market sorted portfolios. We show that our model, calibrated to match the time series properties of aggregate consumption and the stock market, can capture the key properties of the cross section of asset dividends and prices. Quantitatively, the difference in returns on value and growth firms in our calibration averages about 5.5% per year under conventional long-run risk parameters. We also show that, in the model, value assets that have high exposure to long-run consumption risks always have high alphas in the conditional CAPM regressions. In simulations, the model-implied average conditional alpha of the

high book-to-market stocks is significantly positive, whereas the alpha of the growth portfolio is significantly negative, which is consistent with the pattern of CAPM mispricing in the data.<sup>1</sup>

Several empirical observations support our model's economic mechanism. First, as shown in Hansen, Heaton, and Li (2008) and Bansal, Dittmar, and Kiku (2009), among others, value stocks are highly sensitive to low-frequency fluctuations in aggregate consumption. We show that the value premium arises in equilibrium if assets in place are highly exposed to long-run consumption risks. Second, our model rationalizes the importance of long-run consumption risks in explaining cross-sectional differences in the observed risk premia, highlighted recently in Bansal, Dittmar, and Lundblad (2005), Hansen, Heaton, and Li (2008), Kiku (2006), Malloy, Moskowitz, and Vissing-Jorgensen (2009), Bansal, Dittmar, and Kiku (2009), and Bansal, Kiku, and Yaron (2007).

Our paper contributes to the literature that studies the relation between expected returns and firms' investment decisions, such as those by Berk, Green, and Naik (1999), Gomes, Kogan, and Zhang (2003), Carlson, Fisher, and Giammarino (2004), Kyle (2004), Cooper (2006), Gârleanu, Panageas, and Yu (2012), and Novy-Marx (2008). However, our paper differs from the existing research along several dimensions.

First, real option-based models typically imply that growth options are riskier than assets in place, either because the price of the strike asset is assumed to be constant in partial equilibrium settings or because unexercised options expire immediately, or both. In these models, growth options are long positions in assets in place and short positions in a risk-free asset and, therefore, are riskier than value assets. In contrast, in our model, growth options are long lived and compete for capital goods, the scarce resource needed to exercise options. Consequently, they are less risky because the price of the strike asset is endogenously procyclical. This implication is consistent with recent evidence in Kogan and Papanikolaou (2010), who show that, in the data, option-intensive firms generally yield lower returns than firms with higher stocks of physical capital.

Second, to account for the value premium, most of the existing real option-based models require that high book-to-market firms, not growth firms, be option intensive. In our model, value firms

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<sup>1</sup>The failure of the standard market and consumption betas has been illustrated in Mankiw and Shapiro (1986), Fama and French (1992), Lewellen and Nagel (2006), and Petkova and Zhang (2005). The conditional versions of the CAPM, their testable implications, and pertinent econometric issues are studied in Jagannathan and Wang (1996), Ferson and Harvey (1991, 1999), Lettau and Ludvigson (2001), Santos and Veronesi (2006), and Brandt and Chapman (2008), among others.

are assets-in-place intensive while growth firms have higher loadings on options. Therefore, growth firms in our model feature a higher duration of their cash flows relative to value firms. The empirical evidence on growth intensity and duration of book-to-market sorted portfolios suggests that value firms have fewer growth opportunities compared with growth (or low book-to-market) firms. For example, Dechow, Sloan, and Soliman (2004) and Da (2009) show that growth stocks tend to pay off far in the future, and value stocks are characterized by the short duration of their assets. Analysts' forecasts of long-term growth are also systematically lower for value firms than they are for growth firms. In addition, compared with growth firms, in the data high book-to-market firms tend to have a lower ratio of capital expenditure to sales, which is typically viewed as a proxy for growth options available to firms (Da, Guo, and Jagannathan, 2009).

Third, many existing models are built on only one source of risk.<sup>2</sup> Therefore, although they are able to generate a high expected return for value firms, in all these models the conditional CAPM still holds. In contrast, our model is able to account for the failure of the conditional CAPM in the data. We allow for two sources of risks: long- and short-run fluctuations in aggregate consumption. We also consider a representative agent with the recursive preferences, as in Kreps and Porteus (1978), Epstein and Zin (1989), and Weil (1989). Because the two consumption risks in the recursive utility framework carry different risk compensations, a one-factor model such as the CAPM fails to account for the equilibrium asset prices.

Our paper is also related to the broader literature on firms' decisions and their asset pricing implications. McDonald and Siegel (1986) study optimal timing decisions of investment under uncertainty. Abel and Eberly (1994) provide a unified framework of investment under frictions. Abel, Dixit, Eberly, and Pindyck (1996) study the link between the option pricing approach and q-theory of optimal investment. Zhang (2005) and Gala (2005) are recent applications of investment theory to the study of the cross section of equity returns. Gourio (2006) focuses on implications of operating leverage for the value premium.

On a technical level, we obtain closed-form solutions for a general equilibrium model with aggregate uncertainty and a nontrivial cross section of assets. Similar techniques for solving for the cross-sectional distribution of firms have been used by Miao (2005) in a partial equilibrium model

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<sup>2</sup>That is, with the exception of Berk, Green, and Naik (1999) and Kyle (2004), who allow for two sources of risks.

and Luttmer (2007) in a general equilibrium economy without aggregate risks. Optimal stopping problems in economies with long-run risks and recursive preferences are also studied in Bhamra, Kuehn, and Strebulaev (2010) and Chen (2010) in the context of firms' capital structure decisions.

The paper is organized as follows. In Section 2 we present a simple example that illustrates the key mechanism of the model. In Section 3, we set up the model and define the appropriate concept of equilibrium. In Section 4, we solve the model and characterize the cross-sectional distribution of assets. In Section 5, we describe the condition for the value premium to exist in equilibrium and analyze the failure of the conditional CAPM. We calibrate the model and discuss its quantitative implications in Section 6. Section 7 concludes. All technical details and proofs are provided in the Appendix.

## 2. Simple example

Here, we present a simple example that illustrates the basic intuition behind the economic mechanism of our model. We consider an economy with two types of assets: assets in place and blueprints. Assets in place are functioning production units that generate consumption goods. Blueprints represent growth opportunities that, once implemented, result in the creation of new assets in place. Implementing a blueprint requires a unit of capital goods. Therefore, blueprints are options on assets in place and can be replicated by a portfolio with a long position in assets in place and a short position in capital goods. Thus, the riskiness of blueprints depends on the riskiness of assets in place and capital goods. Using a simple set-up, we show that, in equilibrium, the price of capital goods is procyclical and acts as a hedge against risks in assets in place, making growth options less risky relative to assets in place.

Consider an economy with two dates,  $t = 1, 2$ . Aggregate productivity, which we denote as  $\theta_t$ , follows a two-state Markov chain with state space  $\{\theta_H, \theta_L\}$ , where  $\theta_H > \theta_L$ . The transition probabilities are

$$P(\theta_2 = \theta_H | \theta_1 = \theta_H) = P(\theta_2 = \theta_L | \theta_1 = \theta_L) = 1 - p, \quad (1)$$

where  $p \in [0, 1]$ .

At  $t = 1$ , the economy is endowed with measure  $m(\theta_1)$  of blueprints and measure  $\frac{1}{2}m(\theta_1)$  of capital goods. Here, we allow the endowment of blueprints and capital goods to depend on the

aggregate state in an arbitrary way through the  $m(\theta)$  function, but we assume that the ratio of capital goods to blueprints is constant. Each blueprint is characterized by its quality  $Z$ , drawn from a uniform distribution on  $(0, 1)$ . We assume that  $Z$  is orthogonal to the aggregate state  $\theta$  and independent across blueprints. At date 1, a blueprint can be implemented using one unit of capital goods. Implementing a blueprint with quality  $Z$  results in an asset in place that produces  $Z\theta_1$  units of consumption goods at  $t = 1$  and nothing at  $t = 2$  (i.e., we assume that the asset in place evaporates completely in the second period). Alternatively, a blueprint can be used to produce  $\varphi\theta_2$  units of consumption at  $t = 2$  if not implemented. We assume that  $\varphi < \frac{1}{2}$ .

To keep this example simple, we assume that the representative agent is risk-neutral and the interest rate is zero. Although the risks in  $\theta_t$  are not priced, we can still use this economy to examine the sensitivity of asset prices to aggregate risks. In our full model (presented in Section 3), in which the representative agent is risk-averse, differences in exposure to aggregate shocks translate directly into differences in expected returns.

The value of an asset in place equals that of the consumption goods it produces, i.e.,  $Z\theta_1$ . Owners of a blueprint could choose to implement it at date 1 or forgo the opportunity and produce consumption goods at date 2. Let  $q(\theta_1)$  denote the price of capital good. The value of a blueprint with quality  $Z$  is, therefore, given by

$$\max \{ Z\theta_1 - q(\theta_1), \varphi E[\theta_2 | \theta_1] \}. \quad (2)$$

Effectively, blueprints are call options on assets in place with capital goods as the strike asset. Assets in place are risky, because their value,  $Z\theta_1$ , depends on aggregate productivity. The riskiness of growth options is determined by the sensitivity of  $Z\theta_1 - q(\theta_1)$  with respect to  $\theta_1$ . We now turn to the determination of the cost of option exercise  $q(\theta_1)$ .

In equilibrium, the supply of capital goods allows only half of blueprints to be implemented at  $t = 1$ . Competition among blueprints implies that all growth options with  $Z \geq \frac{1}{2}$  are exercised and that the marginal blueprint with  $Z = \frac{1}{2}$  must be indifferent between acquiring a unit of capital goods at time  $t = 1$  or waiting to produce at date 2. Using Eq. (2), this indifference condition can be written as

$$\frac{1}{2}\theta_H - q(\theta_H) = \varphi [(1-p)\theta_H + p\theta_L] \quad (3)$$

and

$$\frac{1}{2}\theta_L - q(\theta_L) = \varphi[(1-p)\theta_L + p\theta_H]. \quad (4)$$

The above equations can be used to solve for the equilibrium price of capital goods:

$$q(\theta_H) = \left(\frac{1}{2} - \varphi\right)\theta_H + \varphi p(\theta_H - \theta_L) \quad (5)$$

and

$$q(\theta_L) = \left(\frac{1}{2} - \varphi\right)\theta_L - \varphi p(\theta_H - \theta_L). \quad (6)$$

It then follows that, for all  $Z \in [0, 1]$ ,

$$\frac{q(\theta_H)}{q(\theta_L)} > \frac{Z\theta_H}{Z\theta_L}. \quad (7)$$

That is, when the economy is hit by a negative productivity shock, the price of capital goods falls by a higher percentage than does the value of assets in place, because capital goods are more sensitive to aggregate risks than are assets in place.

To better understand the intuition behind inequality in Eq. (7), first consider the case in which  $p \rightarrow 0$ . When the probability of a regime switch in  $\theta$  is infinitesimal, the price of capital goods converges to

$$q(\theta_H) = \left(\frac{1}{2} - \varphi\right)\theta_H \quad (8)$$

and

$$q(\theta_L) = \left(\frac{1}{2} - \varphi\right)\theta_L. \quad (9)$$

Therefore,

$$\frac{q(\theta_H)}{q(\theta_L)} = \frac{\theta_H}{\theta_L} = \frac{Z\theta_H}{Z\theta_L}. \quad (10)$$

In this case, capital goods are as risky as assets in place. Because the probability of a regime shift is close to zero, virtually no uncertainty exists about the state of the economy in the next period. Hence, the value of a growth option is linear in the aggregate state  $\theta$  and so is the market clearing price of capital goods.

Suppose now that  $p > 0$  and consider the implication of market clearing on the equilibrium price of capital goods. The payoff of waiting in state  $\theta_H$  is  $\varphi [\theta_H - p(\theta_H - \theta_L)]$ . Given that  $p > 0$ , the possibility of a regime shift in  $\theta$  lowers the benefit of waiting in the high productivity state, because if the shift to  $\theta_L$  does happen, then the value of the blueprint at date 2 falls. If  $q(\theta_H)$  stays the same as in the case of  $p = 0$ , then growth options with quality just below  $\frac{1}{2}$  strictly prefers exercising immediately. To satisfy the indifference condition in Eq. (3),  $q(\theta_H)$  has to increase to deter entrance. By the same logic, the possibility of a regime switch increases the benefit of waiting in the bad state by  $\varphi p(\theta_H - \theta_L)$ , as shown in Eq. (4). To induce entrance in the bad state and restore equilibrium,  $q(\theta_L)$  must decrease. Hence, the market clearing condition requires the price of capital goods to vary more with the aggregate state than the value of assets in place.

Blueprints or growth options are long positions in assets in place and short positions in capital goods. Procyclical variation in the price of capital goods partially offsets risk exposure in assets in place, thus making growth options less risky than value assets. It follows from Eq. (7) that, relative to assets in place, all in-the money options are less sensitive to changes in aggregate productivity, i.e.,

$$\frac{Z\theta_H - q(\theta_H)}{Z\theta_L - q(\theta_L)} < \frac{Z\theta_H}{Z\theta_L}, \quad \text{for all } Z \geq \frac{1}{2}. \quad (11)$$

It can be shown that all out-of-money options also feature lower exposure to risks in  $\theta$  than the value of assets in place.

Two assumptions are essential for growth options to be less risky than assets in place in this example. First, we assume that the aggregate state,  $\theta_t$ , follows a mean-reverting process.<sup>3</sup> Mean reversion creates incentives for the owners of blueprints to expedite option exercise whenever  $\theta_t$  is above its mean. Postponing option exercise in the high state is associated with a potentially lower value of the resulting asset in place, because aggregate productivity is likely to revert to its average. Similarly, an anticipated recovery encourages waiting whenever  $\theta_t$  is below its mean. Thus, mean reversion in the aggregate state introduces cyclical variation in the demand for capital goods.

Second, we assume that blueprints are in excess supply relative to capital goods. In the good state, when option exercise is highly profitable, blueprints compete for the relatively scarce capital,

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<sup>3</sup>As we show below, in our fully dynamic model, this condition can be generalized: Mean reversion around a growing trend suffices to generate an equilibrium value premium.



thus driving up its price. In the bad state, the demand for capital goods is low, and for the market to clear, the price of capital goods has to fall. Thus, the market clearing condition combined with the two assumptions we make are sufficient to generate the procyclical dynamics of the price of capital goods. This procyclicality provides a hedge against risks in assets in place, making growth options less risky.

### 3. The model set-up

In this section, we imbed the above mechanism in a fully dynamic general equilibrium model with long-run risks. This allows us to endogenize some of the assumptions in the above example, quantitatively evaluate the importance of the proposed economic mechanism, and confront the model with the observed cross section of asset dividends and prices.<sup>4</sup>

#### 3.1. Preferences

Consider an economy with a continuum of households that have identical intertemporal preferences described by the Kreps and Porteus (1978) utility with a constant relative risk aversion parameter  $\gamma$ , a constant intertemporal elasticity of substitution (IES)  $\psi$ , and a constant time-discount rate  $\rho$ . Time is continuous and infinite. We follow Duffie and Epstein (1992a, 1992b) and represent the preferences as a stochastic differential utility. Let  $\{U_t\}_{t \geq 0}$  denote the utility process of a representative agent. Given the process for consumption  $\{C_s : s \geq 0\}$ , for every  $t \geq 0$ , the date- $t$  utility of the agent is defined recursively by

$$U_t = E_t \left[ \int_t^\infty F(C_s, U_s) ds \right].^5 \quad (12)$$

In the above equation,  $F(C, U)$  is the aggregator of the recursive preferences, given by

$$F(C, U) = \frac{\rho}{1 - 1/\psi} \frac{C^{1-1/\psi} - ((1 - \gamma)U)^{\frac{1-1/\psi}{1-\gamma}}}{((1 - \gamma)U)^{\frac{1-1/\psi}{1-\gamma} - 1}}.^6 \quad (13)$$

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<sup>4</sup>In the simple example we assume that the value of a blueprint is  $\varphi\theta_2$  if it is not implemented at  $t = 1$ . In our dynamic model, the option value is determined endogenously.

<sup>5</sup>Epstein and Zin (1989) and Weil (1989) develop a general formulation of recursive preferences. Duffie and Epstein (1992b) discuss the representation of stochastic differential utility in continuous time and its extension to the infinite horizon case. Duffie and Lions (1992) and Schroder and Skiadas (1999) study the existence and uniqueness of stochastic differential utility of the Kreps and Porteus (1978) type.

<sup>6</sup>In general, recursive preferences are characterized by a pair of aggregators  $(F, \mathcal{A})$ . Duffie and Epstein (1992b) show that one can always normalize  $\mathcal{A} = 0$ . The aggregator  $F$  used here is the normalized aggregator.

Consistent with the long-run risks literature (Bansal and Yaron, 2004), we assume a preference for early resolution of uncertainty and an IES higher than one, i.e.,  $\gamma > 1 > \frac{1}{\psi}$ .

### 3.2. Outline of the production side

The production side of our economy is composed of production units that yield consumption goods as the output. Production units are assets in place and are built from two types of inputs: a blueprint and one unit of capital goods. Blueprints can be interpreted as innovations and, thus, are subject to idiosyncratic uncertainty. Capital goods represent structures and equipment and are assumed to be homogeneous in quality. A unit of capital goods can be used to implement any innovation, and the productivity of the resulting asset in place depends only on the quality of the embedded innovation, not on that of capital goods. Blueprints and capital goods are supplied exogenously, and we set up the dynamics of the endowments and the production technology in the following subsections.

In our setting, blueprints are effectively growth options that allow their owners to exchange a unit of capital goods (i.e., the strike asset) for a production unit. Blueprints are growth assets. They do not carry any capital goods and feature long duration, in the sense that they do not generate any cash flow immediately but are expected to pay out in the distant future. Production units represent value assets. They carry one unit of capital goods and generate cash flows in the form of consumption goods.

### 3.3. Endowments

The economy has two types of endowments: blueprints and capital goods. Blueprints arrive exogenously at the rate of  $m_t$  per unit of time. The growth rate of  $m_t$  is given by

$$\frac{dm_t}{m_t} = \theta_t dt + \sigma_C(\theta_t) dB_t. \quad (14)$$

In Eq. (14),  $\{B_t\}_{t \geq 0}$  is a one-dimensional standard Brownian motion and  $\{\theta_t\}_{t \geq 0}$  is a two-state Markov process with state space  $\Theta = \{\theta_H, \theta_L\}$ , where  $\theta_H > \theta_L$ .<sup>7</sup> The transition probability of  $\theta_t$

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<sup>7</sup>For simplicity, we assume that the state variable is a two-state Markov chain. Most of the results in the paper remain if we assume that  $\theta$  is a finite state, ergodic Markov chain.

over an infinitesimal time interval  $\Delta$  is given by

$$\begin{bmatrix} 1 - \lambda_H \Delta & \lambda_H \Delta \\ \lambda_L \Delta & 1 - \lambda_L \Delta \end{bmatrix}. \quad (15)$$

As we show later,  $\frac{dm_t}{m_t}$  is the growth rate of the economy along the balanced growth path. Therefore, periods of  $\theta_t = \theta_H$  and  $\theta_t = \theta_L$  can be interpreted as expansions and recessions, respectively. We further assume that  $\sigma_C(\theta_H) \leq \sigma_C(\theta_L)$ , which is consistent with the time series dynamics of the volatility of aggregate consumption growth in the data and allows the model to account for the observed countercyclical variation in risk premia.

The endowment of capital goods arrives exogenously at rate  $\delta m_t$  per unit of time, where  $\delta < 1$ . That is, we assume that the arrival rate of capital goods is proportional to, but less than that of, blueprints at all times.

### 3.4. Technology

Blueprints are storable. A blueprint by itself does not produce any consumption goods but starts production once implemented. Implementing a blueprint requires one unit of capital goods and results in the creation of a production unit.

Blueprints differ in their quality. We assume that the initial quality of all blueprints is a constant  $X_0$ . After birth, a blueprint dies at Poisson rate  $\kappa > 0$ . Conditional on survival, the quality of blueprint  $i$ , denoted by  $X_t^i$ , evolves according to the stochastic differential equation

$$dX_t^i = X_t^i [\mu_O dt + \sigma_O dB_t^i], \quad (16)$$

until the blueprint is either implemented and becomes a production unit or is hit by a Poisson death shock. The  $\{B_t^i\}_{t \geq 0}$  is a standard Brownian motion and is independent across blueprints. The  $\mu_O$  and  $\sigma_O$  are constants. To ensure that the economy has a balanced growth path, we assume that the measure of idle blueprints grows at the same rate as the arrival of new blueprints,  $m_t$ . Taking into account the exogenous death shock, a unit measure of unimplemented blueprints grows by  $\frac{m_{t+s}}{m_t} e^{-\kappa s}$  from time  $t$  to  $t+s$ . This specification can also be interpreted to mean that blueprints can be used either to construct new production units or to produce replicas of themselves.

Capital goods are used to implement blueprints and build production units. They can be stored

and, if not in use, depreciate at rate  $\kappa_C$  per unit of time. Capital goods are homogeneous and can be utilized to implement any blueprint.

A production unit is created by combining one blueprint and one unit of capital goods. A profit-maximizing firm that owns a blueprint can choose to purchase a unit of capital goods and implement the blueprint at any time. Implementation decision is irreversible. Once combined with a blueprint, capital goods cannot be alienated and used productively with a different blueprint at a later date.

The initial amount of consumption goods produced by a production unit is determined by the quality of the carried-out blueprint at the time of implementation. In particular, suppose production unit  $i$  is constructed at time  $\tau$ , and let  $D_t^i$  denote its output at time  $t$ . Then, at time  $t = \tau$ ,

$$D_\tau^i = X_\tau^i, \quad (17)$$

and afterward  $D_t^i$  evolves as

$$dD_t^i = D_t^i \left[ (\mu_A dt + \sigma_A dB_t^i) + \frac{dm_t}{m_t} \right], \quad \forall t > \tau. \quad (18)$$

The output of production units is immediately paid out to shareholders as dividends. We further assume that production units die exogenously at Poisson rate  $\kappa$ . The dividend growth rate of production units is exposed to two sources of risks: the idiosyncratic shock,  $dB_t^i$ , and aggregate risks in  $\frac{dm_t}{m_t}$ .

We assume that blueprints and capital goods are supplied exogenously and that their arrival rates are procyclical. In addition, blueprints are always in excess supply relative to capital goods. These features of our model can arise as an equilibrium outcome if we model the creation of blueprints and capital goods as an optimal response to fundamental technological shocks, as is done in standard real business cycle models. In such a framework, it is typically optimal to create more production units when productivity is high. Because both blueprints and capital goods are necessary investments in building production units, they are likely to inherit the cyclical properties of productivity shocks, as does physical investment in frictionless real business cycle models. Further, we expect blueprints in equilibrium to be in excess supply relative to capital goods due to their option value of waiting. Because blueprints are subject to idiosyncratic shocks,

it is generally optimal to produce a surplus of them, as doing so allows agents to implement higher-quality blueprints first and to delay those with low quality until they receive favorable productivity shocks in the future. In contrast, because capital goods are homogeneous, they do not carry any option value. Therefore, it is not optimal to produce and store idle capital goods.

We focus on the determination of prices of growth options, asset in place, and capital goods in general equilibrium, as well as its implications for the riskiness of growth options relative to assets in place.

### 3.5. Definition of equilibrium

Let  $\{\pi_t\}_{t \geq 0}$  denote the state price density of the economy,  $V_A(D, \theta)$  be the value function of assets in place (production units), and  $V_O(X, \theta)$  denote the value of growth options (blueprints). The value of an asset in place is determined by the discounted stream of future consumption goods it produces:

$$V_A(D_t^i, \theta_t) = E_t \left[ \int_t^\infty \frac{\pi_s}{\pi_t} e^{-\kappa(s-t)} D_s^i ds \right]. \quad (19)$$

Given the dividend process in Eq. (18),  $V_A(D, \theta)$  is homogeneous of degree one in the current level of dividends and can be written as

$$V_A(D_t^i, \theta_t) = a(\theta_t) D_t^i. \quad (20)$$

The price-to-dividend ratio of production units,  $a(\theta_t)$ , is given in Eq. (72) in Appendix Section A.2.

Owners of blueprints must decide on the optimal timing to turn their growth options into assets in place. The optimal stopping problem can be written as

$$V_O(X_t^i, \theta_t) \equiv \max_{\tau} E_t \left[ \frac{\pi_{\tau} m_{\tau}}{\pi_t m_t} e^{-\kappa(\tau-t)} \{V_A(X_{\tau}^i, \theta_{\tau}) - q(\theta_{\tau})\} \right], \quad (21)$$

where  $q(\theta)$  is the equilibrium price of capital goods and the optimization is taken over all stopping times  $\tau$  (adapted to an appropriately defined filtration).

Owners of capital goods can sell their possessions immediately to owners of blueprints who decide to build production units. Alternatively, they can store their capital goods and sell them at a later date. We focus on the class of equilibria in which storage of capital goods is never optimal.

This requires

$$q(\theta_t) \geq E_t \left[ \frac{\pi_s}{\pi_t} e^{-\kappa_C(s-t)} q(\theta_s) \right], \quad \text{for all } s > t. \quad (22)$$

Intuitively, owners of capital goods could have incentives to store them if the price of capital goods is expected to rise in the future. However, storage cannot be optimal if either the discount rate or the depreciation of capital goods is relatively high. In fact, given the parameter values of the model, there is a lower bound on the depreciation rate,  $\kappa_C^*$ , such that inequality (22) holds as long as  $\kappa_C \geq \kappa_C^*$ . We give the expression for  $\kappa_C^*$  in Appendix Section A.2. We assess the magnitude of the model-implied depreciation threshold in the calibration section.

We focus on the balanced growth path of the economy, where all equilibrium quantities grow at the same rate  $\frac{dm_t}{m_t}$ . Later, we verify that the balanced growth path exists. Blueprints enter and exit the economy continuously. Some of them leave because they receive good productivity shocks and exercise their options to become production units; others die due to the exogenous Poisson shock. Along the balanced growth path, the total rate of exit must be the same as the rate of entry,  $m_t$ . In fact, as we show in Appendix Section A.3, after being normalized by  $m_t$ , the cross-sectional distribution of the quality of options is time-invariant. We use  $\Phi(\cdot)$  to denote its density. With this notation,  $\Phi(X) \times m_t$  is the density of unexercised options with quality  $X$ .

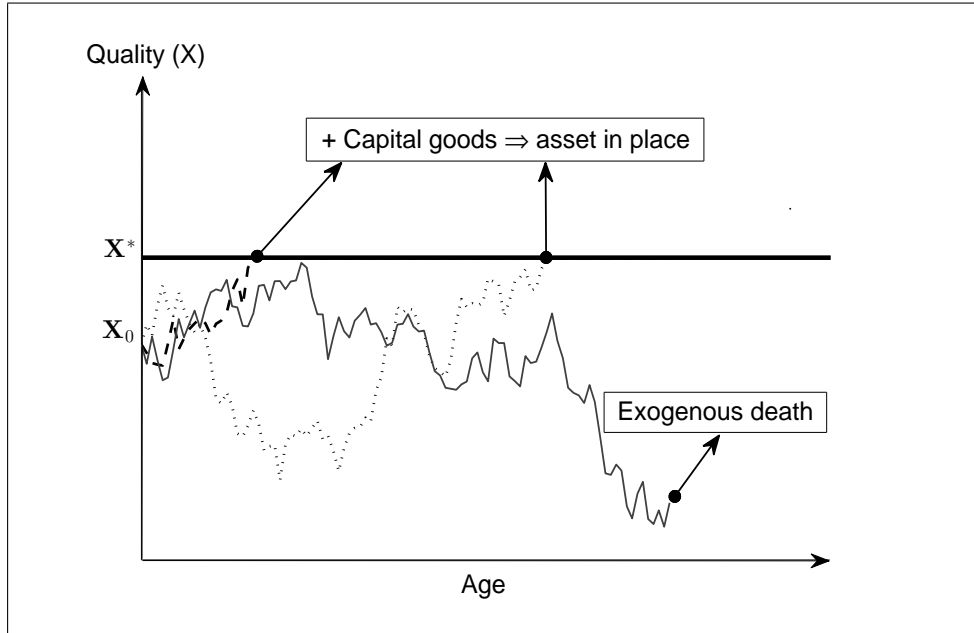
The balanced growth path of the economy is characterized by a constant option exercise threshold  $X^*$  such that an option is exercised if, and only if, its quality is higher than  $X^*$ . Intuitively, in any efficient equilibrium, blueprints with higher quality must be implemented first. Therefore, there is a cutoff level of the quality of blueprints,  $X^*(t)$ , potentially time-varying, such that at each date  $t$  only blueprints with quality higher than the cutoff level are implemented. Because options become assets in place once their quality reaches the threshold,  $X^*(t)$  is the absorbing barrier of the cross-sectional distribution of the quality of options. We use the notation  $m_{EXIT}[\Phi, X^*(t)]$  to denote the absorbing rate of density  $\Phi$  at the absorbing barrier  $X^*(t)$ . That is,  $m_{EXIT}[\Phi, X^*(t)]$  is the measure of blueprints that cross over the exercise threshold level  $X^*(t)$  per unit of time. Because all quantities in the economy grow at the common rate  $m_t$  along the balanced growth path, the rate of option exercise at time  $t$  is  $m_{EXIT}[\Phi, X^*(t)] \times m_t$ . Market clearing requires that the rate of option exercise be equal to the rate of arrival of capital goods,  $\delta m_t$ . Therefore, the

option exercise threshold  $X^*(t)$  must satisfy

$$m_{EXIT}[\Phi, X^*(t)] = \delta. \quad (23)$$

Because  $\Phi$  is a stationary distribution, Eq. (23) implies that  $X^*(t)$  must be constant along the balanced growth path. Hence, we denote it by  $X^*$ .

Fig. 1 depicts the dynamics of a cohort of newly arrived growth options. At time  $t$ , a measure  $m_t$  of blueprints with initial quality  $X_0$  arrives. Some of them reach the threshold level  $X^*$  in the future and become assets in place. The sample paths of two such blueprints are represented by the two dashed lines. Blueprints can also die prematurely because of the Poisson death shock, which is shown by the solid line. Along the balanced growth path, the total measure of blueprints that are implemented at time  $t$  is  $\delta \times m_t$ , and the total measure of those that are hit by the Poisson death shock is  $(1 - \delta) \times m_t$ .



**Fig. 1.** Dynamics of growth options. This figure illustrates the dynamics of a cohort of growth options.  $X_0$  represents the initial quality of a newly born option.  $X^*$  is the equilibrium option-exercise threshold. Options whose quality reaches  $X^*$  obtain a unit of capital goods and become assets in place (dashed and dotted lines, respectively); others exit the economy once they are hit by the exogenous death shock (solid line).

In our set-up, blueprints and capital goods are indivisible and must be used in a one-to-one ratio. Therefore, the implementation of blueprints is a discrete choice, and the equilibrium features

a cutoff rule,  $X^*$ . This specification is consistent with the empirical fact that investment is lumpy at the plant level and implies that there is always a nontrivial measure of unimplemented blueprints in equilibrium. Indivisibility disappears at the aggregate level, because each blueprint and production unit is of measure zero. Consequently, aggregate quantities in our model are smooth, as in the data.

**Definition 1.** *A competitive equilibrium with balanced growth is a collection of equilibrium prices and quantities that satisfies the following conditions.*

(a) *Shareholder value maximization for growth options: The option exercise threshold  $X^*$  solves the optimal stopping problem in Eq. (21).*

(b) *Optimality of nonstorage of capital goods: The inequality in Eq. (22) holds for all  $t$ .*

(c) *Market clearing for capital goods: The rate of option exercise equals the rate of the arrival of capital goods as in Eq. (23).*

(d) *Market clearing for consumption goods: Aggregate consumption is the sum of the output of all production units. Let  $\mathcal{I}_t$  denote the set of production units that are active at time  $t$ ; the aggregate consumption of the economy at time  $t$  is given by*

$$C_t = \int_{i \in \mathcal{I}_t} D_t^i di. \quad (24)$$

(e) *Consistency of macro- and micro-variables: The normalized density  $\Phi$  is consistent with the law of motion of the quality of individual growth options in Eq. (16).*

Condition (e) requires that the variables describing the macroeconomic quantities be consistent with the individual behavior of growth options. Technically, this requirement implies that  $\Phi$  has to satisfy a version of the Komogorov forward equation. We provide a further discussion in Appendix Section A.3.

#### 4. Characterization of equilibrium

In this section, we construct a competitive equilibrium with balanced growth defined above. We first conjecture that the equilibrium aggregate consumption is proportional to the arrival of new blueprints. This implies that aggregate consumption growth rate is  $\frac{dm_t}{m_t}$  and allows us to



solve the optimal stopping problem of growth options and derive the cross-sectional distribution of blueprints and production units. We then impose the market clearing condition to solve for the equilibrium price of capital,  $q(\theta)$ . Finally, we verify our conjecture and show that the total output of all production units adds up to aggregate consumption.

#### 4.1. The optimal stopping problem

We conjecture that, along the balanced growth path, aggregate consumption grows at the same rate as  $m_t$ . Thus, fluctuations in consumption growth are driven by the time-varying expected growth component,  $\theta_t$ , and transient shocks,  $dB_t$ . Therefore, our model endogenously generates long-run predictable variations in consumption growth as assumed in Bansal and Yaron (2004). Following the long-run risk literature, we call risks in  $\theta_t$  long-run risks and use short-run risks to refer to the iid component of consumption growth.

**Conjecture 1.** *Aggregate consumption,  $C_t$ , evolves as*

$$\frac{dC_t}{C_t} = \theta_t dt + \sigma_C(\theta_t) dB_t. \quad (25)$$

Given the law of motion of aggregate consumption, we can determine the state price density of the economy as in the Lucas (1978)–Breeden (1979) framework. The functional form of the pricing kernel  $\{\pi_t\}_{t \geq 0}$  is given in Section A.1 of the Appendix. We make the following assumption on the parameters of the model to guarantee that the values of growth options and assets in place are finite:

$$\rho - \left(1 - \frac{1}{\psi}\right)\theta + \frac{1}{2}\gamma \left(1 - \frac{1}{\psi}\right)\sigma_C^2(\theta) + \kappa - \mu > 0, \quad \text{for } \theta = \theta_H, \theta_L, \text{ and } \mu = \mu_O, \mu_A. \quad (26)$$

Under the assumption in Eq. (26), the optimal stopping problem of blueprints has a well-defined solution for any  $q(\theta)$ . In general, given any functional form of the price of capital,  $q(\theta)$ , we can summarize the solution to the optimal stopping problem of growth options by a pair of thresholds,  $X^*(\theta_H)$  and  $X^*(\theta_L)$ , such that it is optimal to exercise a growth option in state  $\theta$  if  $X_t$  reaches  $X^*(\theta)$  from below, for  $\theta = \theta_H, \theta_L$ . The solution to the optimal stopping problem is provided in Proposition 1.

**Proposition 1.** *Given the equilibrium price of capital,  $q(\theta)$ , the value function of growth options is given by*

$$V_O(X, \theta_H) = K_1 e_1 X^{\zeta_1} + K_2 e_2 X^{\zeta_2}, \quad V_O(X, \theta_L) = K_1 X^{\zeta_1} + K_2 X^{\zeta_2}. \quad (27)$$

The parameters  $1 < \zeta_1 < \zeta_2$ , and  $e_1 > 0$ ,  $e_2 < 0$  are described in the Appendix Subsection A.2.2.

The option exercise thresholds,  $X^*(\theta_H)$  and  $X^*(\theta_L)$ , along with the two constants  $K_1, K_2$ , are jointly determined by the following value-matching and smooth-pasting conditions:

$$\begin{bmatrix} V_O(X^*(\theta_H), \theta_H) \\ V_O(X^*(\theta_L), \theta_L) \end{bmatrix} = \begin{bmatrix} V_A(X^*(\theta_H), \theta_H) - q(\theta_H) \\ V_A(X^*(\theta_L), \theta_L) - q(\theta_L) \end{bmatrix} \quad (28)$$

and

$$\begin{bmatrix} \frac{\partial V_O}{\partial X}(X^*(\theta_H), \theta_H) \\ \frac{\partial V_O}{\partial X}(X^*(\theta_L), \theta_L) \end{bmatrix} = \begin{bmatrix} \frac{\partial V_A}{\partial X}(X^*(\theta_H), \theta_H) \\ \frac{\partial V_A}{\partial X}(X^*(\theta_L), \theta_L) \end{bmatrix}. \quad (29)$$

Proof: See Appendix, Section A.2 ■

Balanced growth implies that  $X^*(\theta_H) = X^*(\theta_L) = X^*$ . This condition helps to determine the market clearing price of capital.

#### 4.2. Distribution of growth options and assets in place

Along the balanced growth path, after being normalized by the common trend  $m_t$ , the cross-sectional distribution of the quality of blueprints is time-invariant. Appendix Section A.3 provides closed-form solutions for the invariant distribution  $\Phi$  and the exit rate of growth options for any constant option exercise threshold,  $X^*$ . In Appendix Section A.3, we also show that the rate of option exercise is strictly decreasing in  $X^*$ . Intuitively, the higher the level of  $X^*$ , the harder it is for options to reach the threshold. Therefore, fewer options are exercised per unit of time. In equilibrium, market clearing requires the rate of option exercise be equal to the rate of arrival of capital goods. Consequently, a unique level of  $X^*$  exists consistent with the stationary distribution  $\Phi$ . Proposition 2 provides a closed form solution for the equilibrium market clearing  $X^*$  along the balanced growth path. The equilibrium prices and quantities are completely characterized once we determine the market clearing  $X^*$ .

**Proposition 2.** *The unique option exercise threshold  $X^*$  determined by the market clearing condition in Eq. (23) is given by*

$$X^* = \delta^{\frac{1}{\eta_2}} X_0, \quad (30)$$

where  $\eta_2$  is a time-invariant constant defined as

$$\eta_2 = \left( \frac{\mu_O}{\sigma_O^2} - \frac{1}{2} \right) - \sqrt{\left( \frac{\mu_O}{\sigma_O^2} - \frac{1}{2} \right)^2 + \frac{2\kappa}{\sigma_O^2}} < 0. \quad (31)$$

Proof: See Appendix, Section A.3 ■

Given  $X^*$ , we use Proposition 1 to solve for the equilibrium price of capital  $q(\theta)$  and the value function of growth options. Once we obtain the equilibrium prices, we verify that the storage of capital goods is not optimal and, therefore, does not occur in equilibrium. This property of the equilibrium requires that the depreciation rate of capital goods exceeds a given threshold level  $\kappa_C^*$ . We provide a formal proof of this result in Appendix Subsection A.2.3. We close the model by verifying Conjecture 1, that aggregate consumption grows at the same rate as  $m_t$ . We elaborate the details of the construction of the balanced growth path in Appendix Section A.3.

## 5. Asset pricing implications

To highlight the asset pricing implications of the model, we first derive a general formula for the equilibrium risk premium and establish conditions under which options are less risky than assets in place. We then show that assets in place are characterized not only by high risk premia but also high alphas in the conditional CAPM regressions.

### 5.1. Risk premium

Let  $R_{t,t+\Delta}^i$  denote the return of asset  $i$  over the time interval  $[t, t + \Delta]$ . Taking into account the rate of growth of unexercised options, the realized return of option  $i$ , conditional on survival, is given by

$$R_{t,t+\Delta}^i = \frac{m_{t+\Delta} V_O(X_{t+\Delta}^i, \theta_{t+\Delta})}{m_t V_O(X_t^i, \theta_t)}. \quad (32)$$

Similarly, if spared the Poisson death shock during the interval  $[t, t + \Delta]$ , the realized return of asset in place  $i$  is

$$R_{t,t+\Delta}^i = \frac{V_A(D_{t+\Delta}^i, \theta_{t+\Delta}) + D_t^i \Delta}{V_A(D_t^i, \theta_t)}. \quad (33)$$

The risk premium of asset  $i$  at time  $t$ , which we denote by  $RP(i, t)$ , is defined as

$$RP(i, t) = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} E_t [R_{t,t+\Delta}^i - r(\theta_t) \Delta], \quad (34)$$

where  $r(\theta_t)$  is the instantaneous risk-free rate of the economy.

Because compensation for risks in  $\theta$  (i.e., long-run risks) plays an important role in our analysis, it is convenient to introduce the transition probability of  $\theta$  over an infinitesimal interval  $\Delta$  under the risk-neutral measure

$$\begin{bmatrix} 1 - \hat{\omega}^{-1} \lambda_H \Delta & \hat{\omega}^{-1} \lambda_H \Delta \\ \hat{\omega} \lambda_L \Delta & 1 - \hat{\omega} \lambda_L \Delta \end{bmatrix}, \quad (35)$$

where  $\hat{\omega}$  is given in Eq. (67) in Appendix Section A.1. The transition probabilities in Eq. (35) differ from the physical probabilities in Eq. (15) only by a factor of  $\hat{\omega}$ . Under preference for early resolution of uncertainty,  $\hat{\omega} < 1$ . Consequently, the probability of downturns (regime switches from  $\theta_H$  to  $\theta_L$ ) are amplified under the risk-neutral measure, i.e.,  $\hat{\omega}^{-1} \lambda_H \Delta > \lambda_H \Delta$ . Similarly, the risk-neutral measure deflates the probability of recoveries from  $\theta_L$  to  $\theta_H$ :  $\hat{\omega} \lambda_L \Delta < \lambda_L \Delta$ . Therefore, under our parameterizations of preferences, long-run risks in  $\theta$  require a positive risk premium.

To simplify notation, we denote the price of asset  $i$  at time  $t$  as  $V(i, t, \theta_t)$ , with the understanding that  $V(i, t, \theta_t) = V_O(X_t^i, \theta_t)$ , if asset  $i$  is a growth option, and  $V(i, t, \theta_t) = V_A(D_t^i, \theta_t)$ , if asset  $i$  is an asset in place. The risk premium of a generic asset  $i$  is given in Proposition 3.

**Proposition 3.** *The risk premium of asset  $i$  is*

$$RP(i, t) = \gamma \sigma_C^2(\theta_L) + \lambda_L (1 - \hat{\omega}) \left[ \frac{V(i, t, \theta_H)}{V(i, t, \theta_L)} - 1 \right], \quad \text{if } \theta_t = \theta_L \quad (36)$$

and

$$RP(i, t) = \gamma \sigma_C^2(\theta_H) + \lambda_H (1 - \hat{\omega}^{-1}) \left[ \frac{V(i, t, \theta_L)}{V(i, t, \theta_H)} - 1 \right], \quad \text{if } \theta_t = \theta_H. \quad (37)$$

Proof: See Appendix, Section A.4 ■

The risk premium of an asset consists of two components. The first term can be written as

$$\gamma\sigma_C^2(\theta_t) = \gamma \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \text{Cov}_t \left( R_{t,t+\Delta}^i, \frac{C_{t+\Delta} - C_t}{C_t} \right), \quad (38)$$

which is the premium for the covariation of the asset return with contemporaneous innovations in consumption growth, i.e., the compensation for the asset's exposure to short-run consumption risks. Proposition 3 implies that assets in place and growth options have the same exposure to short-run risks and, consequently, carry identical short-run risk premia.

The second component of the risk premium is the compensation for the covariation with innovations in the expected consumption growth,  $\theta_t$ , i.e., compensation for long-run risks. We refer to this as the long-run risk premium. If there is a regime switch from  $\theta_L$  to  $\theta_H$ , then the net return on asset  $i$  is  $\frac{V(i,t,\theta_H)}{V(i,t,\theta_L)} - 1$ .  $\lambda_L \hat{\omega}$  is the risk-neutral probability of transition from  $\theta_L$  to  $\theta_H$ . It follows that the long-run risk premium is the difference between the expected return under the physical measure,  $\lambda_L \left[ \frac{V(i,t,\theta_H)}{V(i,t,\theta_L)} - 1 \right]$ , and that under the risk-neutral measure,  $\lambda_L \hat{\omega} \left[ \frac{V(i,t,\theta_H)}{V(i,t,\theta_L)} - 1 \right]$ . The expression for the risk premium in state  $\theta_H$  has a similar interpretation.

Also, given  $\hat{\omega} < 1$ , it follows from Proposition 3 that the long-run risk premium of asset  $i$  is higher than that of asset  $j$  in both states of the world if

$$\frac{V(i,t,\theta_H)}{V(i,t,\theta_L)} > \frac{V(j,t,\theta_H)}{V(j,t,\theta_L)}. \quad (39)$$

Intuitively, the larger the change in asset value in case of a regime switch in the expected consumption growth, the higher the exposure to long-run risks and, therefore, the higher the compensation for risks in  $\theta_t$ . In the following subsection, we discuss conditions under which value assets have higher exposure to long run risks than growth assets.

## 5.2. Value premium

Proposition 4 provides a sufficient condition under which growth options are less risky than assets in place.

**Proposition 4.** *Under the assumptions of the model, and the technical condition  $\mu_A > \bar{\mu}$  ( $\bar{\mu}$  is given in Appendix Section A.4), all assets in place have higher exposure to long-run risks than do*

*growth options:*

$$\frac{V_A(D, \theta_H)}{V_A(D, \theta_L)} > \frac{V_O(X, \theta_H)}{V_O(X, \theta_L)}, \quad \text{for all } D \text{ and } X. \quad (40)$$

*Hence, assets in place carry higher risk premia than do growth options.*

Proof: See Appendix, Section A.4 ■

Because growth options can be represented as portfolios with a long position in assets in place and a short position in capital goods, for options to have low exposure to long-run risks and, therefore, carry a low premium, the price of capital goods (the strike asset) must be procyclical. If this is true, then risks in the price of capital goods and assets in place partly offset each other, making growth options less risky than value assets.

Two assumptions of the model ensure the procyclicality of the price of capital goods: (1) mean reversion in aggregate risks and (2) the scarcity of capital goods relative to growth options. Mean reversion provides incentives for option exercise in the good state and discourages it in the low-growth state, thus generating a cyclical variation in the demand for capital goods. As long as the supply of capital goods is not fully elastic, their price must absorb fluctuations on the demand side and, therefore, it features procyclical dynamics. To better understand these implications, consider a blueprint at the option exercise threshold  $X^*$  and suppose that the economy is currently in the high-growth state ( $\theta = \theta_H$ ). Exercising the option right away creates an asset in place with a value of  $a(\theta_H) X^*$ . Waiting, however, is associated with a nontrivial probability of a regime switch in  $\theta$  and, consequently, a potential drop in the value of the asset in place to  $a(\theta_L) X^*$  [note that  $a(\theta_L) < a(\theta_H)$ ]. Thus, mean reversion in aggregate state, i.e., a possibility of a regime shift, encourages option exercise in the good state and creates high competition for capital goods among option owners. Given that capital goods are in relatively scarce supply, their price must increase for markets to clear. Similarly, in recessions, option owners have incentives to postpone exercising their options until the economy recovers. Therefore, the price of capital goods has to fall to make option exercise attractive enough and restore the equilibrium.<sup>8</sup>

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<sup>8</sup>To further understand the importance of aggregate mean-reversion, note that all assets in the model have the same exposure to short-run risks,  $dB_t$ . This is because the latter are iid and, therefore, do not affect the relative risk exposure of growth options and assets in place.

Although the two assumptions are sufficient to generate procyclical dynamics of the price of capital goods, the technical condition in Proposition 4 ensures that cyclical variations in the price are strong enough for the value premium to exist. Quantitatively, for growth options to be less risky relative to value assets, the price of capital goods must be more sensitive to long-run risks than is the value of assets in place. The technical condition,  $\mu_A > \bar{\mu}$ , guarantees this result. In words, it requires dividends of individual assets in place to have higher exposure to long-run risks than aggregate consumption.<sup>9</sup> This condition is consistent with empirical evidence in the long-run risk literature. In the data, value firms' cash flows respond strongly to permanent or near-permanent consumption shocks (see, for example, Bansal, Dittmar, and Lundblad, 2005; and Hansen, Heaton, and Li, 2008). In addition, empirical research shows that value assets have higher exposure to long-run consumption risks than do growth assets (Bansal, Dittmar, and Lundblad, 2005; Hansen, Heaton, and Li, 2008; Kiku, 2006; Malloy, Moskowitz, and Vissing-Jorgensen, 2009; Bansal, Kiku, and Yaron, 2007; and Bansal, Dittmar, and Kiku, 2009). Our model provides an endogenous link between these empirical observations: A high exposure of cash flows of assets in place to persistent consumption risks translates into a relatively low risk exposure of growth options through the equilibrium adjustment of the price of capital goods.

We illustrate the mechanism of our model in Fig. 2.<sup>10</sup> We use dashed lines to represent value functions of assets in place in the high state (thick line) and in the low state (thin line). Solid lines are value functions of growth options. At the option exercise threshold  $X^*$ , the distance between the value of assets in place and that of growth options equals the price of capital goods in the corresponding state. As the economy shifts from one state to another, the price of capital goods changes by more, percentage-wise, than does the value of assets in place, i.e., the price of capital goods is more sensitive to long-run risks relative to value assets. Thus, a portfolio with a long

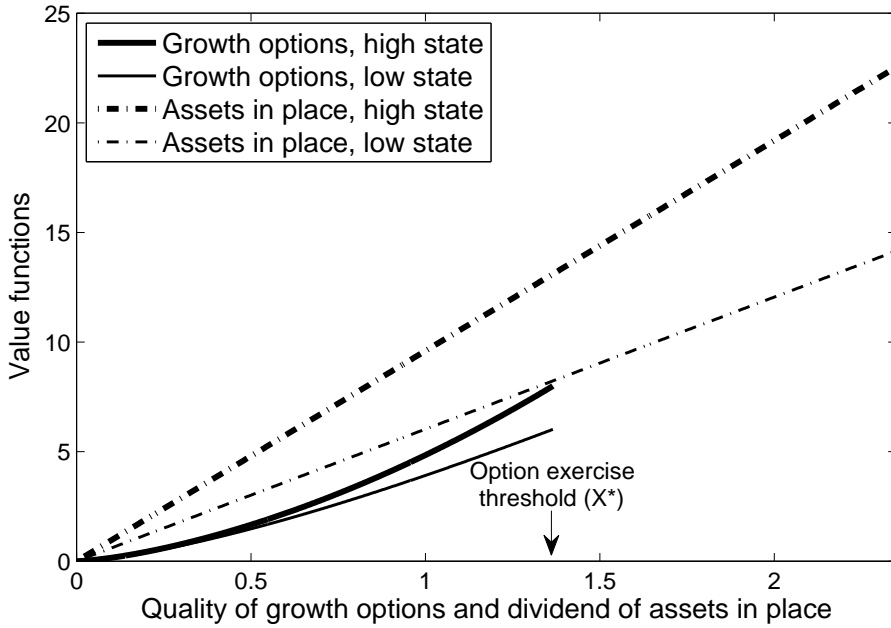
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<sup>9</sup>Although dividends of all production units sum up to aggregate consumption according to the general equilibrium restriction, in our set-up the dividends of individual assets in place are allowed to be more or less risky than aggregate consumption. The expected growth rate of individual dividends differs from that of aggregate consumption by  $\mu_A$  [see Eq. (18)]. An increase in  $\mu_A$ , for example, boosts the dividend growth rate of individual production units and the level of aggregate consumption but leaves unaffected the growth rate of the economy along the balanced growth path. At the same time, a higher  $\mu_A$  amplifies dividend exposure to long-run risks relative to that of aggregate consumption.

If we relax the assumption of general equilibrium and model aggregate dividends separately from aggregate consumption, then Proposition 4 and all the model implications hold under similar general conditions. The proof of claims for the partial equilibrium set-up is available upon request.

<sup>10</sup>Figs. 2 and 3 are constructed under the parameter values calibrated in Section 6.

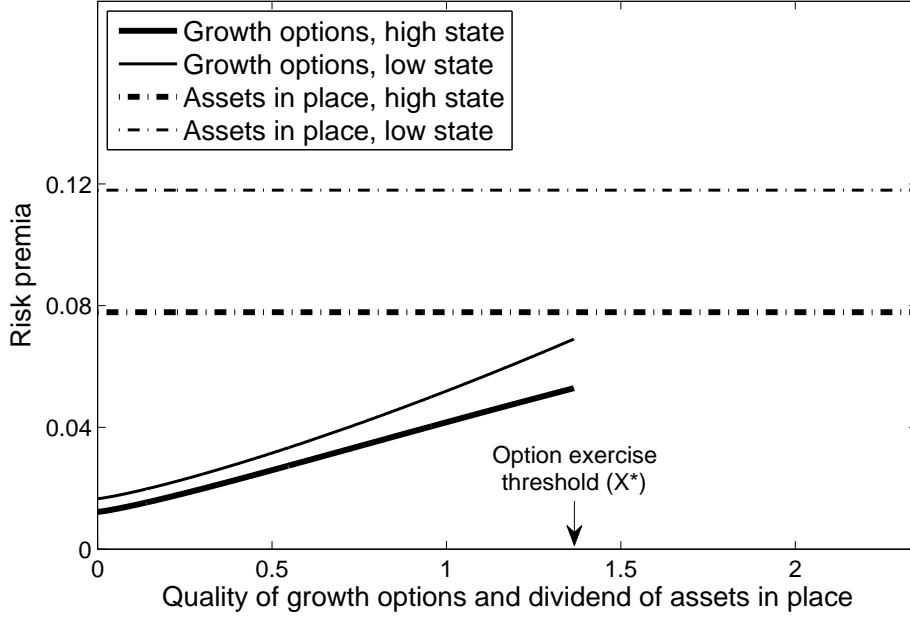
position in value assets and a short position in capital goods is less risky than assets in place, because the procyclicality of the long and short positions partially cancel out one another. In our model, growth options are precisely such portfolios. As the figure shows, the value of growth options responds less to a shift in  $\theta_t$  compared with assets in place. In Fig. 3 we plot the model-implied risk premia of growth options and assets in place against option quality and dividends of assets in place. The dashed lines denote the risk premia of assets in place, and the solid lines represent the risk premia of growth options. As the figure shows, uniformly, value assets are riskier and carry a higher risk premium than do growth options.



**Fig. 2.** Value functions of growth options and assets in place. This figure plots the value functions of growth options and assets in place for the two states, high ( $\theta_H$ ) and low ( $\theta_L$ ). At the option-exercise threshold,  $X^*$ , the distances between the values of assets in place (dashed lines) and those of options (solid lines) are the equilibrium prices of capital goods.

The proposed mechanism is absent in the real option-based models of Berk, Green, and Naik (1999) and Gomes, Kogan, and Zhang (2003), because they assume that options depreciate fully if not exercised immediately. Consequently, there are no unexercised options in their equilibria. In Carlson, Fisher, and Giammarino (2004) and Gârleanu, Panageas, and Yu (2012), the supply of the resources needed for option exercise is assumed to be infinitely elastic. Thus, the presence of unexercised options in these models has no pricing implications for the cost of exercise, and the





**Fig. 3.** Risk premia of growth options and assets in place. This figure plots the risk premia of growth options against their quality and the risk premia of assets in place as a function of their dividend for the two states, high ( $\theta_H$ ) and low ( $\theta_L$ ). Risk premia are expressed in annual percentage terms.

latter is always risk-free. In contrast, in our model, growth options are always in excess supply in equilibrium and have to compete for capital goods, a scarce resource that is needed for option exercise. As we show above, this competition in aggregate risks results in the equilibrium value premium.

For tractability, we assume an exogenous law of motion of capital goods. More generally, we expect the basic intuition of our model to hold if capital goods are produced endogenously and are installed with convex adjustment costs. In high-growth states, capital goods become more valuable; hence, it is optimal to increase their production. Therefore, the arrival rate of capital goods is higher when  $\theta_t = \theta_H$  (just as in our model). At the same time, due to installation costs, the marginal cost of capital goods is also higher in good times. As a result, changes in the quantity of capital goods do not fully absorb productivity shocks in  $\theta_t$ , and the equilibrium features both a higher arrival rate of capital goods and a higher price of capital goods.

### 5.3. Failure of conditional CAPM

To understand the CAPM implications of the model, suppose an econometrician observes a return series that is generated under the null of our model and aims to test the pricing restrictions implied by the standard CAPM. To draw inferences about the conditional CAPM, she runs the following regression:

$$R_{t,t+\Delta}^i - r(\theta_t) \Delta = \alpha_{t,\Delta}^i + (R_{t,t+\Delta}^W - r(\theta_t) \Delta) \beta_{t,\Delta}^i + \varepsilon_{t,t+\Delta}, \quad (41)$$

where  $R_{t,t+\Delta}^i$  is the rate of return of asset  $i$  over  $[t, t + \Delta]$ ,  $R_{t,t+\Delta}^W$  is the return of aggregate wealth, and  $r(\theta_t) \Delta$  is the risk-free rate over the same time interval.

If the CAPM were true, then all the alphas in specification (41) would be zero. This could happen only if the aggregate wealth portfolio were perfectly correlated with the true stochastic discount factor (SDF). In our model, innovations in the SDF are driven by both long-run and short-run risks in consumption. Importantly, the two risks carry very different risk premia. A one-factor model such as the CAPM, in general, is not able to convey all the necessary pricing information to account for the cross-sectional differences in expected returns. To formalize this claim, consider theoretical values of the CAPM  $\alpha$  and  $\beta$ :

$$\alpha_{t,\Delta}^i = (E_t [R_{t,t+\Delta}^i] - r(\theta_t) \Delta) - \beta_{t,\Delta}^i (E_t [R_{t,t+\Delta}^W] - r(\theta_t) \Delta) \quad (42)$$

and

$$\beta_{t,\Delta}^i = \frac{Cov_t \left( R_{t,t+\Delta}^i, R_{t,t+\Delta}^W \right)}{Var_t \left( R_{t,t+\Delta}^W \right)}. \quad (43)$$

Using Proposition 3, we obtain the following characterization of the CAPM alphas in our model economy.<sup>11</sup>

**Proposition 5.** *Assets with high exposure to long-run risks obtain high alphas in the conditional CAPM.*

Proof: See Appendix, Section A.4 ■

Given that value assets are more sensitive to long-run risks than are growth options, they feature

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<sup>11</sup>Appendix Section A.4 contains the population values for the CAPM alphas.

higher alphas in the conditional CAPM regressions. To better understand this result, notice first that the SDF in our economy is driven by consumption growth and the return on the wealth portfolio. Given that the two are not perfectly correlated, the CAPM factor does not trace all the movements of the pricing kernel. Further, under preferences for early resolution of uncertainty, the exposure of the stochastic discount factor to long-run risks is larger than the corresponding exposure of the wealth portfolio. Therefore, value assets that carry a high long-run risk premium in the model are overpriced by the standard CAPM. We assess the magnitude of alphas in both conditional and unconditional CAPM regressions in our calibration exercise below.

#### 5.4. Duration of growth and value assets

Due to differences in the timing of cash flows, growth options feature longer cash flow duration than do assets in place. This implication is intuitive. Value assets are already established production units that generate consumption goods. These are already implemented blueprints that have exhausted their growth opportunities, and the only event that they face going forward is the exogenous exit from the market. Thus, assets in place derive their value mostly from current and near-future dividends and have a relatively short duration. Unexercised options or blueprints have an option to grow. They are potential, or delayed-into-the-future, production units with a cash flow stream that is shifted more toward future dates. Hence, growth options have longer duration of their cash flows compared with value assets.

To quantify differences in duration of value and growth assets, we use the Macaulay's measure of duration. In particular, for an asset with dividend process  $\{D_t : t \geq 0\}$ , its duration at time  $t$  is defined as

$$E_t \left[ \int_0^\infty s \times \frac{\pi_{t+s}}{\pi_t} \frac{D_{t+s}}{p_t} ds \right], \quad (44)$$

where  $p_t$  is the value of the asset that satisfies

$$p_t = E_t \left[ \int_0^\infty \frac{\pi_{t+s}}{\pi_t} D_{t+s} ds \right]. \quad (45)$$

In our model, the duration of assets in place depends only on the aggregate state variable  $\theta_t$ , which we denote by  $M_A(\theta)$ . The expressions for  $M_A(\theta_H)$  and  $M_A(\theta_L)$  are given in the Appendix, Section A.5. We denote the duration of growth options as  $M_O(X, \theta)$ , because it depends on both the aggregate state,  $\theta_t$ , and the quality of the blueprint,  $X_t$ . Proposition 6 provides closed-form

solutions for duration of growth options.

**Proposition 6.** *The Macaulay duration of growth options is given by*

$$M_O(\theta, X) = \frac{Q(\theta, X)}{V_O(\theta, X)}, \quad (46)$$

where

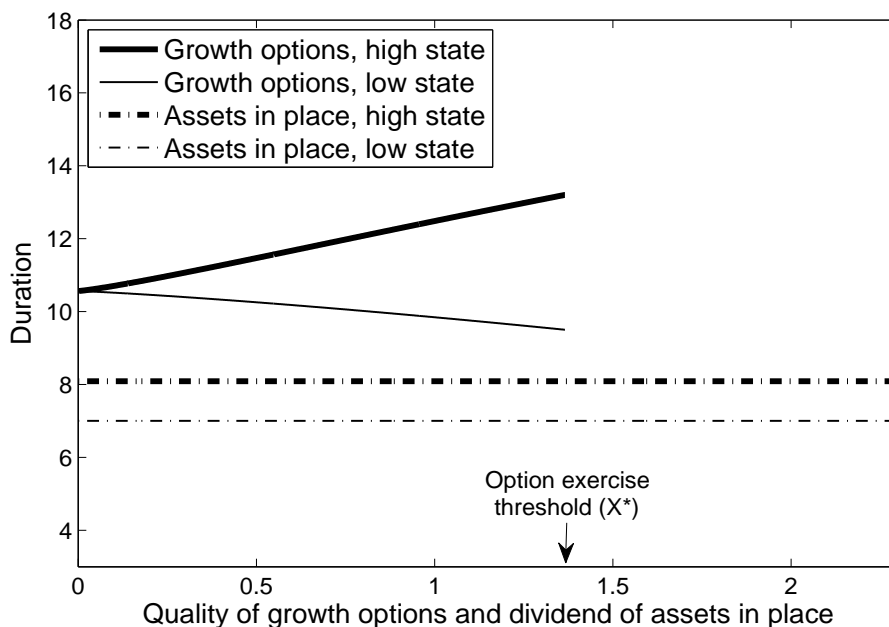
$$Q(\theta_H, X) = L_1 e_1 X^{\zeta_1} + L_2 e_2 X^{\zeta_2}, \quad Q(\theta_L, X) = L_1 X^{\zeta_1} + L_2 X^{\zeta_2}. \quad (47)$$

The parameters  $1 < \zeta_1 < \zeta_2$  and  $e_1 > 0$ ,  $e_2 < 0$  are defined in Proposition 1, and  $L_1$  and  $L_2$  are given in the Appendix Section A.5.

Proof: See Appendix, Section A.5 ■

We illustrate the model-implied differences in duration of growth and value assets in Fig. 4, which is constructed under the parameter configuration that we use in our calibration section below. The figure shows that the cash flow duration is substantially shorter for value assets than it is for growth options. Further, the duration of assets in place is longer in the high state of the economy, due to better growth expectations and lower discount rates. Averaging across the two regimes, the effective maturity of value assets in the model is about 7.5 years. The duration of growth options is at least three to four years longer and, depending on the state of the economy, is either increasing or decreasing in option quality. In the high state, the duration of a growth option is monotonically increasing in its moneyness. When  $\theta_t = \theta_H$ , options that are close to the exercise threshold have the longest duration, because they are likely to be implemented in the high-growth and low-discount-rate state. As the quality of option deteriorates, the time-to-exercise lengthens and the likelihood of being exercised in the high state declines due to mean reversion in aggregate risk. Consequently, the duration of the option becomes shorter. The opposite happens in the low-growth state. When  $\theta_t = \theta_L$ , deep out-of-the-money options have a longer duration than do at-the-money blueprints, because they face a longer waiting period and, therefore, are more likely to be implemented after the economy shifts to the high-growth state. Although empirical estimates of duration are too scarce and assumption-driven to make any direct comparison of the model with the data meaningful, the model-implied magnitudes seem economically reasonable and

are generally comparable to the available estimates [for example, those reported in Dechow, Sloan, and Soliman (2004)].



**Fig. 4.** Duration of growth options and assets in place. This figure plots Macaulay duration of growth options and assets in place as a function of their quality and dividend, respectively, for the two states, high ( $\theta_H$ ) and low ( $\theta_L$ ). Duration is expressed in years.

A longer duration of growth options relative to assets in place is a feature shared by all real option-based models. The difference between our model and most of the existing explanations lies in the implied relation between book-to-market characteristics and duration. In models in which growth options are riskier than assets in place (for example, in Berk, Green, and Naik, 1999; Gomes, Kogan, and Zhang, 2003; and Carlson, Fisher, and Giammarino, 2004), a value premium is generated by making value firms load more heavily on options compared with growth firms. This class of models implies a positive relation between book-to-market and cash flow duration and, hence, a longer duration of value firms than growth firms. In contrast, in our model, growth firms are option-intensive and the market value of high book-to-market firms is mostly driven by assets in place. Therefore, our model can simultaneously account for the value premium and the inverse relation between book-to-market characteristics and duration shown in Da (2009) and Dechow, Sloan, and Soliman (2004).

## 6. Quantitative evaluation of the model

To evaluate the ability of the model to account for the observed value premium and other features of book-to-market sorted portfolios, we run a simulation exercise. We choose the model parameters to match key properties of aggregate consumption, the stock market index, and the risk-free rate. When the calibrated model ensures reasonable dynamics of aggregate quantities, we examine its performance in the cross section. Our calibration is guided solely by time series dynamics of the observed aggregate data and does not exploit any cross-sectional information.

We simulate the model on a monthly frequency, and we target the dynamics of annual data. To avoid any seasonal and measurement biases in the data, we focus on annual moments. We simulate monthly series over 80 years, aggregate the simulated variables to the annual frequency, and report various moments of the resulting annual data. To remove the effect of initial conditions, we effectively simulate 160 years of data and discard the first half of the sample. We find that increasing the size of the initial simulated sample does not alter the results. We repeat simulations one hundred times and report the medians of various statistics of interest across simulations.

### 6.1. Data sources

Our targeted data in calibration consist of real per capita consumption of nondurables and services; the stock market index of the NYSE, Amex, and Nasdaq traded firms; and the three-month Treasury bill. We obtain consumption data from the National Income and Product Accounts tables published by the Bureau of Economic Analysis (BEA). The stock market and risk-free rate data come from the Center for Research in Security Prices (CRSP). Cross-sectional data that we use at the evaluation stage are made up of three book-to-market sorted portfolios: “Growth” and “Value” portfolios that consist of the firms in the lowest and highest 30th percentile, respectively, and firms in the middle portfolio that we label “Neutral.” The breakpoints are at the 30th and 70th percentiles of the book-to-market sort of the NYSE-listed stocks. Our portfolio construction follows the standard procedure of Fama and French (1992) by using the data from the Compustat and CRSP databases. For each book-to-market portfolio and for the aggregate stock market index, we construct value-weighted returns and per share dividend series as in Campbell and Shiller (1988) and Bansal, Dittmar, and Lundblad (2005). We convert asset data to real by using the personal consumption deflator. All data are annual and span the period from 1930 to 2007.

## 6.2. Parameter configuration and targeted moments

In Table 1 we present our benchmark preference and time series configuration. We choose preference parameters as in the long-run risk literature. Similar to Bansal and Yaron (2004), we set  $\rho$  at 0.01, use a risk-aversion parameter of ten, and set the elasticity of intertemporal substitution at 1.5. This choice of preferences, and the technology parameters discussed below, allows the model to match the dynamics and the level of the risk-free rate, as well as the magnitude of the risk premium in the economy. We also discuss sensitivity of the model's implications to the magnitude of risk aversion and IES.

**Table 1**

Model parameters.

This table presents preferences and technology parameters that we use in calibrating the model. All parameters are expressed in annual terms.

Parameter	Value
<u>Preference</u>	
$\beta$	0.01
$\gamma$	10
$\psi$	1.5
<u>Consumption</u>	
$\theta_H$	0.028
$\theta_L$	-0.012
$\lambda_H$	0.08
$\lambda_L$	0.26
$\sigma_C(\theta_H)$	0.018
$\sigma_C(\theta_L)$	0.026
<u>Cash flows</u>	
$\mu_A(\theta_H)$	0.124
$\mu_A(\theta_L)$	-0.11
$\sigma_A(\theta_H)$	0.35
$\sigma_A(\theta_L)$	0.45

To match the observed consumption and dividend dynamics, we relax the general equilibrium restriction that the total dividend equals aggregate consumption and calibrate them separately. We

continue to assume that aggregate consumption obeys the law of motion described in Eq. (25), but now we use a more flexible parameterization of the dividend dynamics. In particular, we assume that the cash flow generated by asset in place  $i$  follows

$$\frac{dD_t^i}{D_t^i} = \mu_A(\theta_t) dt + \sigma_C(\theta_t) dB_t + \sigma_A(\theta_t) dB_t^i. \quad (48)$$

In this specification, the first two terms govern the dividend growth exposure to long- and short-run consumption risks, respectively, and the last component represents an idiosyncratic shock.<sup>12</sup>

We choose parameters of consumption growth process to match time series dynamics of the consumption data. Table 2 compares the moments of the simulated growth rates with the corresponding sample statistics. Point estimates along with the Newey and West (1987) standard errors are presented in the “Data” column; model-implied statistics, in the “Model” column. Our calibration matches the mean, volatility, and first two autocorrelations of consumption growth. For example, the model-implied first-order autocorrelation of consumption growth is 0.42, which agrees well with the observed persistence of 0.44. In addition to the four moments reported in the table, our calibration is designed to capture evidence on the National Bureau of Economic Research (NBER)-dated business cycle fluctuations. We allow for expansions to last about three times longer than recessions. We define expansions in the model by states with a high mean and a low volatility of consumption growth. Similarly, recessions are associated with both low growth and high uncertainty about future consumption. Thus, our calibration also accounts for a negative covariation between growth and uncertainty observed in the data.<sup>13</sup> In addition, our calibration matches the low-frequency dynamics of the volatility of consumption growth. We estimate consumption volatility by a moving average of absolute residuals from an AR(1) fitted to consumption growth rates. We use a three-year moving average to separate the long-run component from transitory movements in economic uncertainty. The extracted low-frequency volatility component in the data is fairly persistent, with a first-order autocorrelation of 0.68. Our calibration leads to a similar persistence of about 0.67.

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<sup>12</sup>The model solution in this case follows closely that of the general equilibrium set-up and is available upon request.

<sup>13</sup>In the data, consumption uncertainty is persistently high when the economy is in contraction. For instance, the average absolute AR(1)-residual of consumption growth is about 0.02 during the NBER-defined recessions and only 0.01 when the economy expands. We account for this empirical evidence by allowing for high consumption volatility in low states and low volatility when consumption growth is expected to rise.



**Table 2**

Consumption growth dynamics.

This table presents moments of annual consumption growth in the data and in the model.  $E[\Delta c]$  denotes the unconditional mean of consumption growth;  $\sigma(\Delta c)$  is the standard deviation; and  $AC(1)$  and  $AC(2)$  are the first- and second-order autocorrelations of growth rates, respectively. Means and volatilities are expressed in percentage terms. Consumption data, taken from the Bureau of Economic Analysis, are real, annual, per capita series of nondurable expenditures and services from 1930 to 2007. Standard errors of the sample statistics, reported in parentheses, are calculated by using the Newey and West (1987) variance-covariance estimator with four lags. The model-implied moments represent the medians of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency. Each simulation is 80 years long.

Statistic	Data	Model
$E[\Delta c]$	1.94 (0.32)	1.82
$\sigma(\Delta c)$	2.14 (0.52)	2.56
$AC(1)$	0.44 (0.11)	0.42
$AC(2)$	0.16 (0.22)	0.17

We choose option quality and cash flow parameters to match the key moments of dividend growth rates of the aggregate stock market portfolio. In addition to the time series parameters specified in Table 1, we set  $X_0$  at one, which is a pure normalization that has no qualitative or quantitative effect, and use  $\delta = 0.8$ , which implies that 80% of new growth options eventually obtain capital goods and become assets in place and the remaining options die prematurely. We set the annualized values of  $\mu_O$  and  $\sigma_O$  to 0.04 and 0.40, respectively, and assume that options and assets in place depreciate at an annual rate of 10%, i.e.,  $\kappa = 0.1$ . We find that the model implications are generally robust to the choice of parameters that govern the entry and the evolution of unexercised growth options. The lower bound on the depreciation rate of capital implied by our calibration is around 11% per annum, which is well within the range of the existing estimates. For example, Epstein and Denny (1980) and Bischoff and Kokkelenberg (1987) estimate the depreciation rate of physical capital for the US manufacturing sector to be between 10% and 14%. Similar rates of annual depreciation are typically assumed in the production-based literature that explicitly specifies capital accumulation dynamics.

Table 3 presents the implications of the model for the dynamics of the aggregate market

portfolio.<sup>14</sup> Our calibration matched the mean growth rate of aggregate dividends, their volatility, and their correlation with consumption growth. The only dimension that the model has some difficulty addressing is the persistence of dividend growth. In the data, the first-order autocorrelation of dividend growth rates is 0.22, but the corresponding statistic in the model is about 0.58. Although undesirable, in our view, this implication is not critical. The issue of a relatively high serial correlation of dividend growth rates induced by the channel of long-run risks can easily be resolved by introducing another common (but orthogonal to consumption risks) component into firms' cash flows, as in Bansal and Yaron (2004). We do not entertain such an extension, as it has no effect on prices.

The bottom panel of Table 3 shows that the model based on long-run growth risks can successfully account for the historically high equity premium, as has been highlighted in Bansal and Yaron (2004). The model-implied average excess return of the market portfolio is about 6%. As in the data, the correlation between equity returns and consumption growth is low. The model also correctly predicts higher volatility of asset prices relative to dividends. The standard deviation of the market return is about 20% in the data and 22% in the model. The model generates plausible dynamics for the risk-free rate with a mean of 1.7% and a standard deviation of 1%.

In the model, asset prices are strongly procyclical. The correlation between the log of the market-to-book ratio and consumption growth is about 42%. This implication confirms the cyclical properties of the observed series, because in the data, this correlation is approximately 30%. In addition, the model-implied market-to-book ratio is highly persistent, with a first-order autocorrelation of about 0.89, which provides a good match for the sample persistence of 0.84.

### *6.3. Cross section of dividends and returns*

By relying on the assumed time series dynamics and model solutions, we simulate a pool of growth options and assets in place, which we use as building blocks for creating firms. We view a firm as a collection of growth options and assets in place that we randomly sample from the simulated pool. Guided by the cross-sectional dispersion of sales and market capitalization in the data, we assume that the initial distribution of growth options and assets in place across firms is

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<sup>14</sup>As in the data, the market portfolio in simulations is a value-weighted combination of individual firms. We discuss the construction of the cross section in Subsection 6.3.

**Table 3**

Dynamics of aggregate market portfolio.

This table presents data- and model-based moments of annual dividend growth rates and returns of the aggregate stock market portfolio.  $E[\cdot]$  and  $\sigma(\cdot)$  denote the unconditional mean and standard deviation, respectively.  $AC(1)$  is the first-order autocorrelation coefficient, and  $Corr(\cdot, \Delta c)$  denotes the unconditional correlation between the corresponding variable and consumption growth. Means and volatilities are expressed in percentage terms. The aggregate stock market index is the value-weighted portfolio of firms traded on the NYSE, Amex, and Nasdaq. All data are annual, expressed in real terms, and cover the period from 1930 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated by using the Newey and West (1987) variance-covariance estimator with four lags. The model-implied moments represent the medians of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency. Each simulation is 80 years long. The market portfolio in the model is constructed from a simulated pool of two thousand firms.

Statistic	Data	Model
<u>Dividend growth</u>		
$E[\Delta d]$	0.81 (1.48)	0.62
$\sigma(\Delta d)$	11.99 (2.48)	11.02
$AC(1)$	0.22 (0.14)	0.58
$Corr(\Delta d, \Delta c)$	0.66 (0.23)	0.63
<u>Return</u>		
$E[R]$	8.46 (1.88)	7.61
$\sigma(R)$	19.52 (1.98)	22.10
$AC(1)$	-0.03 (0.11)	-0.02
$Corr(R, \Delta c)$	0.07 (0.15)	0.09

Pareto. After stapling growth options and assets in place, we track each firm over time, replacing extinct units with brand-new growth options.<sup>15</sup> In each simulation, the size of the cross section is equal to two thousand firms. We sort the simulated sample of firms into three book-to-market portfolios following the same sorting procedure as in the data.

Table 4 illustrates the dynamics of the per share dividend growth rates of the resulting book-to-market sorted portfolios and compares them with the data. The value portfolio in the data

<sup>15</sup>Although the number of firms in our simulation is fixed, the composition of growth options and assets in place inside each firm changes over time. We experimented with a more complex way of creating a cross section, with the number of firms increasing over time as in the data. This alternative simulation requires additional assumptions about firms' entrance but does not materially affect the dynamics of the resulting portfolios.

is characterized by high unconditional growth, and firms in the growth portfolio on average exhibit low per share growth. Our model is able to replicate this feature of the data. As the book-to-market ratio increases, the unconditional growth of the per share dividends raises from 0.45% to about 4.70% in the data, and from -0.36% to about 3.61% in the model. The model-implied volatilities of growth rates are also similar to their data counterparts, except for the highest book-to-market portfolio. However, this discrepancy speaks in favor of, rather than against, the model, because the high volatility of the growth rates in the value portfolio is driven by few, virtually zero, dividend observations in the beginning of the sample, when the data quality is somewhat suspect. Further, the model-implied correlations between portfolio growth rates and aggregate consumption are all within 1 standard error from the data statistics and have a similar, hump-shaped cross-sectional pattern. Overall, our calibration seems to capture the key moments of the cross section of dividend growth rates reasonably well, although we do not choose the model parameters to target any moments of the cross-sectional data.

From the perspective of our model, a more interesting dynamic characteristic of cash flows is their low-frequency (rather than contemporaneous) covariation with consumption. Empirically, the exposure of dividend growth rates to long-run consumption risks is increasing from a low to a high book-to-market portfolio, as shown in Bansal, Dittmar, and Lundblad (2005), Bansal, Dittmar, and Kiku (2009), and Hansen, Heaton, and Li (2008). Our model features a similar positive relation between long-run cash flow betas and book-to-market characteristics. Because in simulations the long-run risk variable is conveniently available, we measure long-run exposures inside the model by regressing monthly dividend growth rates onto the expected growth component,  $\theta_t$ . We find a monotonically increasing pattern in the model-implied long-run betas, starting from 4.5 for the growth portfolio and reaching 8.2 for the value portfolio.

Table 5 presents the average returns for the simulated portfolios along with their empirical counterparts. Consistent with the data, the model generates a sizable variation in the risk premium that is increasing in book-to-market characteristic. The model-implied mean return of growth firms is 5.86%, and the average compensation for holding value firms is 11.34%. Thus, the model-implied value premium is about 5.5% per annum, which quantitatively matches the difference between mean returns on high and low book-to-market portfolios in the data.

**Table 4**

Dividend growth rate dynamics of book-to-market portfolios.

This table presents the sample and model-implied means ( $E[\Delta d]$ ) and standard deviations ( $\sigma(\Delta d)$ ) of the annual per share dividend growth rates for book-to-market sorted portfolios and their correlations with aggregate consumption growth ( $Corr(\Delta d, \Delta c)$ ). Means and volatilities are expressed in percentage terms. The “Growth” and “Value” portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. The “Neutral” portfolio contains firms in the middle of the distribution. All data are annual, expressed in real terms, and cover the period from 1930 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated by using the Newey and West (1987) variance-covariance estimator with four lags. The model-implied moments represent the medians of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency, each simulation is 80 years long, and the size of the simulated cross section is two thousand firms.

Portfolio	Data	Model
<u><math>E[\Delta d]</math></u>		
Growth	0.45 (1.40)	-0.36
Neutral	1.98 (1.63)	1.28
Value	4.70 (2.30)	3.61
<u><math>\sigma(\Delta d)</math></u>		
Growth	13.67 (2.10)	12.12
Neutral	14.58 (3.85)	9.80
Value	21.24 (4.59)	10.58
<u><math>Corr(\Delta d, \Delta c)</math></u>		
Growth	0.40 (0.16)	0.41
Neutral	0.66 (0.23)	0.71
Value	0.55 (0.20)	0.68

As in Bansal and Yaron (2004), we allow for time variation in the conditional volatility of consumption growth and, therefore, variation in the risk premium. The model-implied spread in expected returns on value and growth firms also exhibits countercyclical dynamics. Inside the model, the correlation between the conditional value premium and the volatility of consumption growth is 0.18, and its correlation with expected growth in consumption is -0.34. To compute these numbers, we construct the conditional value premium by regressing the spread in realized annual returns of value and growth portfolios on their lagged price-dividend ratios. We measure the expected growth in consumption by fitting an AR(1) process to annual consumption growth and

**Table 5**

Average returns of book-to-market portfolios.

This table presents the average annual returns on book-to-market sorted portfolios in the data and in the model. The “Growth” and “Value” portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. The “Neutral” portfolio contains firms in the middle of the distribution. All data are annual, expressed in real percentage terms, and cover the period from 1930 to 2007. Standard errors of the data statistics, reported in parentheses, are calculated by using the Newey and West (1987) variance-covariance estimator with four lags. The model-implied moments represent the medians of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency, each simulation is 80 years long, and the size of the simulated cross section is two thousand firms.

Portfolio	Data	Model
Growth	7.75 (1.94)	5.86
Neutral	9.47 (1.99)	9.28
Value	13.21 (2.21)	11.34

construct its volatility by taking a three-year moving average of absolute residuals from the above regression. The countercyclical dispersion in expected returns of value and growth firms implied by the model is consistent with empirical evidence in Kiku (2006) and Chen, Petkova, and Zhang (2008).

#### 6.4. CAPM implications

In the model, as in the data, the standard CAPM fails. We illustrate the magnitude of the model deviations from the unconditional and conditional CAPM predictions in Tables 6 and 7.

Table 6 reports the unconditional CAPM alphas and the corresponding  $t$ -statistics for each of the book-to-market portfolios. The unconditional CAPM inside the model is strongly rejected. All three alphas are economically and statistically significant. Similar to the data, the CAPM tends to underprice growth stocks (by about 2.3% per annum) and overprice value stocks (by about 4.6%). Both the pattern and the magnitude of simulated alphas are consistent with the CAPM mispricing in the data. In addition, the model-implied CAPM betas display a negative relation with average returns. For example, the low-premium growth portfolio has a beta of about 1.08, and the high-premium value portfolio has a beta of 0.85.

The conditional CAPM also falls short in explaining the cross-sectional dispersion in risk premia inside the model. To evaluate the performance of the conditional market betas, we run

**Table 6**

Unconditional capital asset pricing model (CAPM) alphas of book-to-market portfolios.

This table illustrates the performance of the unconditional CAPM in the data and in the model. The entries represent intercepts and the corresponding  $t$ -statistics from an ordinary least squares regression of annual excess returns for book-to-market portfolios on the excess return of the aggregate stock market. The “Growth” and “Value” portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. The “Neutral” portfolio contains firms in the middle of the distribution. The observed aggregate stock market index is the value-weighted portfolio of firms traded on the NYSE, Amex, and Nasdaq. The risk-free rate in the data is measured by using the three-month Treasury bill. The data set covers the period from 1930 to 2007. The model-implied moments represent the medians of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency, each simulation is 80 years long, and the size of the simulated cross section is two thousand firms.

Portfolio	Data		Model	
	$\alpha$	T-statistic	$\alpha$	T-statistic
Growth	-0.34	-0.64	-2.27	-3.32
Neutral	0.74	0.77	2.56	2.18
Value	2.82	2.09	4.64	4.31

**Table 7**

Average conditional capital asset pricing model (CAPM) alphas of book-to-market portfolios.

This table illustrates the performance of the conditional CAPM in the data and in the model. The entries are average (annualized) alphas and their  $t$ -statistics from the 36-month rolling regressions of excess returns for book-to-market portfolios on the excess return of the aggregate stock market. The “Growth” and “Value” portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. The “Neutral” portfolio contains firms in the middle of the distribution. The observed aggregate stock market index is the value-weighted portfolio of firms traded on the NYSE, Amex, and Nasdaq. The risk-free rate in the data is measured by using the three-month Treasury bill. The data set covers the period from 1930 to 2007. The model-implied moments represent the medians of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency, each simulation is 80 years long, and the size of the simulated cross section is two thousand firms.

Portfolio	Data		Model	
	$\alpha$	T-statistic	$\alpha$	T-statistic
Growth	-0.73	-2.07	-2.89	-4.29
Neutral	1.21	2.89	2.74	2.31
Value	3.36	3.78	4.79	5.60

three-year rolling window regressions of the monthly excess returns of book-to-market portfolios on the monthly excess returns of the aggregate stock market. In Table 7 we present the average alphas and their robust  $t$ -statistics. On average, the model-implied conditional alphas monotonically increase, from about -2.9% for growth firms to almost 4.8% for the value portfolio, thus replicating an empirically strong positive relation between alphas and book-to-market characteristics. Quantitatively, the average conditional alphas in simulations conform to the failure of the conditional CAPM in the actual data. To understand the failure of the standard CAPM, recall that in our model the market return does not serve as a proxy for the true stochastic discount factor and, consequently, cannot correctly price any other asset, as highlighted in Proposition 5.

### *6.5. Additional cross-sectional characteristics*

Panel B of Table 8 summarizes the transitional dynamics of the simulated cross section for our benchmark calibration. For comparison and completeness, in Panel A we report transition frequencies across book-to-market portfolios in the data. We define transition probabilities by the fraction of firms that migrate from one bin in the current year to another bin in the next year we rebalance portfolios. The frequencies that we report in the table are time series averages of transition probabilities.<sup>16</sup> It is apparent that, in the model, firms are likely to stay in their current bin. For example, 83% of firms in the growth portfolio are still classified as growth firms the following year, and 80% of value firms remain in the top book-to-market bin in the next year. The corresponding numbers in the data are 78% and 82% for growth and value portfolios, respectively. Consistent with the data, the model-implied transitions to nearby portfolios are more frequent than is migration to more distant portfolios. We also find that if we compute the transition probabilities as a percentage of market capitalization instead of a percentage of firms, then firms' migrations across portfolios inside the model are similar to the observed transition probabilities.

Table 9 compares the average market shares of book-to-market portfolios between the model and the data. Market shares represent a fraction of the market value of a given portfolio in the total market capitalization. The table shows that the model-implied distribution of market values across portfolios matches well the observed pattern in market shares. In the model, the growth portfolio

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<sup>16</sup>In the data, unlike the model, a fraction of firms exits the market every period. We ignore firms that disappear in the observed sample and rescale all the transition probabilities so that they sum to one. Doing so has only a minor effect on the sample statistics but facilitates the comparison between the model and the data.



**Table 8**

Transition probabilities across book-to-market portfolios.

This table reports average frequencies of transition across book-to-market portfolios in the data (Panel A) and in the model (Panel B). We measure transition probabilities by the fraction of firms that migrate from one bin in year  $t$  to another bin in year  $t + 1$ . The “Growth” and “Value” portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. The “Neutral” portfolio contains firms in the middle of the distribution. The observed data cover the period from 1930 to 2007. The model-implied moments represent the means of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency, each simulation is 80 years long, and the size of the simulated cross section is two thousand firms.

From	To		
	Growth	Neutral	Value
<i>Panel A: Data</i>			
Growth	0.78	0.20	0.02
Neutral	0.15	0.65	0.20
Value	0.01	0.17	0.82
<i>Panel B: Model</i>			
Growth	0.83	0.11	0.06
Neutral	0.06	0.80	0.14
Value	0.05	0.15	0.80

contributes about 55% to the total market portfolio; in the data, its share fluctuates around 54%. The mean share of the value portfolio is 11% in the model and 12% in the data. Consistent with the data, the model-implied shares are persistent. The cross-sectional average of the first-order autocorrelations in market shares is about 0.85.

To summarize, we show that our theoretical model calibrated to match the observed time series dynamics of aggregate consumption and the stock market is able to generate a cross section of firms that is consistent with key properties of the observed book-to-market portfolios and is able to simultaneously reconcile the value premium and empirical failure of the standard CAPM.

### 6.6. Sensitivity to preference parameters

Although preference parameters play an important role in accounting for the level and time series dynamics of asset prices, as long as agents have preferences for early resolution of uncertainty (i.e.,  $\gamma > 1/\psi$ ), the cross-sectional implications of our model are qualitatively robust to the choice

**Table 9**

Market shares of book-to-market portfolios.

This table reports average market shares of book-to-market portfolios in the simulated and observed data. The “Growth” and “Value” portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. The “Neutral” portfolio contains firms in the middle of the distribution. The observed data cover the period from 1930 to 2007. The model-implied moments represent the means of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency, each simulation is 80 years long, and the size of the simulated cross section is two thousand firms.

Portfolio	Data	Model
Growth	0.54	0.55
Neutral	0.34	0.34
Value	0.12	0.11

of risk aversion and IES. Table 10 illustrates the effect of preferences on the level of the risk-free rate and equity return and for the performance of the CAPM.

**Table 10**

Asset pricing implications under alternative preferences.

This table presents the asset pricing implications of the model under alternative values of risk aversion (RA) and intertemporal elasticity of substitution (IES).  $\bar{R}$  and  $\bar{\alpha}$  denote the average return on an asset and the average conditional capital asset pricing model alpha, respectively. “Risk-free” corresponds to a riskless asset. “Market” represents the value-weighted portfolio of all firms; and “Growth” and “Value” portfolios represent firms in the lowest and highest 30th percentile of the book-to-market sort, respectively. The model-implied moments are the medians of the corresponding statistics across one hundred simulations. The model is simulated on a monthly frequency, each simulation is 80 years long, and the size of the simulated cross section is two thousand firms.

Asset	RA = 10, IES = 0.5		RA = 5, IES = 1.5		RA = 2, IES = 1.5	
	$\bar{R}$	$\bar{\alpha}$ (T-statistic)	$\bar{R}$	$\bar{\alpha}$ (T-statistic)	$\bar{R}$	$\bar{\alpha}$ (T-statistic)
Risk-free	5.27		1.70		1.95	
Market	8.29		3.98		2.96	
Growth	6.55	-2.31 (-3.75)	2.79	-1.83 (-3.29)	1.99	-1.67 (-3.08)
Value	11.30	3.39 ( 4.93)	6.52	3.14 ( 4.09)	4.17	2.05 ( 3.61)

In the left panel, we keep risk aversion at its baseline value of ten but set IES at 0.5. The table shows that this specification fails to match the dynamics of interest rates, instead generating a large, volatile risk-free rate with a mean of more than 5% per annum and a standard deviation of

about 4%. In the middle and right panels, we assume that IES is above one but lower the degree of risk aversion to five and two, respectively. A decrease in risk aversion results in lower equity returns and premia on all assets in the economy. However, the key cross-sectional results remain unchanged in all three cases: Value firms carry a substantially higher premium than do growth firms, and the CAPM fails to account for the difference in risk compensations. For example, even when we set risk aversion at two and the model considerably understates the level of the market return, it still generates a significant value premium of about 2.2% and a significantly positive (negative) alpha of the value (growth) portfolio in the conditional CAPM regressions. In all the cases considered, the unconditional CAPM is also unable to fairly price the cross section.

## 7. Conclusion

We present a general equilibrium model of option exercise that provides a rational resolution of the value premium puzzle. Growth options are less risky than assets in place in our model because the equilibrium price of capital goods, physical resource needed for exercising options, is procyclical. The key conditions that ensure this result are mean reversion in aggregate risk and scarcity of the supply of capital goods. Intuitively, in good times, exercising an option is especially valuable because doing so results in the creation of a highly productive value asset. Hence, more options compete for scarce capital, thus driving up its price. Similarly, assets in place are less profitable in bad times and option owners have incentives to delay implementation of their options until the economy recovers. To encourage option exercise and to clear the market, the price of capital goods in recessions has to decrease. Thus, in equilibrium, the price of capital goods is highly procyclical and acts as a hedge against risks in assets in place, making growth options less risky.

To facilitate closed-form solutions and obtain sharp characterizations of the equilibrium, we assume that growth options and capital goods arrive exogenously. Ai (2010) and Ai, Croce, and Li (2011) present general equilibrium models in which growth options and capital goods are produced endogenously as optimal responses to technological shocks as in the real business cycle literature. Findings in these papers provide additional support to our model. They show that options being less risky than assets in place is consistent with the basic properties of the dynamics of macroeconomic quantities such as consumption, investments, and hours worked.

To match key properties of aggregate cash flows and returns, we consider an economy with recursive preferences and long-run risks. We calibrate the model by using time series data on US consumption and stock market index and show that our model can quantitatively account for the cross-sectional dispersion in mean returns of value and growth firms. In addition, our model replicates the failure of the conditional and unconditional CAPM regressions, as well as other stylized features of book-to-market sorted portfolios.

## Appendix

### A.1. State price density

It is convenient to represent the Markov chain  $\{\theta_t\}_{t \geq 0}$  as stochastic integrals with respect to Poisson processes. In particular, let  $\{N_{H,t}\}_{t \geq 0}$  be a Poisson process with intensity  $\lambda_H$  and  $\{N_{L,t}\}_{t \geq 0}$  be an independent Poisson process with intensity  $\lambda_L$ . Let  $I_{\{x\}}$  be the indicator function, that is,

$$I_{\{x\}}(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}. \quad (49)$$

Then  $\{\theta_t\}_{t \geq 0}$  can be represented as

$$d\theta_t = (\theta_H - \theta_L) \times \eta(\theta_t^-)^T dN_t, \quad (50)$$

where  $\eta(\theta)$  and  $N_t$  are vector notations:

$$\eta(\theta) = [-I_{\{\theta_H\}}(\theta), I_{\{\theta_L\}}(\theta)]^T \quad (51)$$

and

$$N_t = [N_{Ht}, N_{Lt}]^T. \quad (52)$$

Here we adopt the convention that  $\{\theta_t\}$  is right-continuous with left limit and use the notation

$$\theta_t^- = \lim_{s \rightarrow t, s < t} \theta_s. \quad (53)$$

We conjecture that the equilibrium consumption of the representative agent satisfies Eq. (25). To guarantee that the life-time utility of the representative agent is finite, we assume

$$\rho + \frac{1}{2}\gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2(\theta) - \left(1 - \frac{1}{\psi}\right) \theta > 0, \quad \text{for } \theta = \theta_H, \theta_L. \quad (54)$$

To solve for the pricing kernel of the economy, we first need to derive the equilibrium utility process of the representative agent. This is given by Lemma 1.

**Lemma 1.** *The utility function of the representative agent is given by*

$$U_t = \frac{1}{1-\gamma} H(\theta_t) C_t^{1-\gamma}. \quad (55)$$

The function  $H(\theta)$  in the above equation is defined as

$$H(\theta_H) = \left\{ \frac{1}{\rho} \left[ \rho + \frac{1}{2}\gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta_H) - \left( 1 - \frac{1}{\psi} \right) \theta_H - \frac{1-1/\psi}{1-\gamma} \lambda_H (\omega^{-1} - 1) \right] \right\}^{-\frac{1-\gamma}{1-1/\psi}} \quad (56)$$

and

$$H(\theta_L) = \left\{ \frac{1}{\rho} \left[ \rho + \frac{1}{2}\gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta_L) - \left( 1 - \frac{1}{\psi} \right) \theta_L - \frac{1-1/\psi}{1-\gamma} \lambda_L (\omega - 1) \right] \right\}^{-\frac{1-\gamma}{1-1/\psi}}, \quad (57)$$

where  $\omega < 1$  is the unique solution to the following equation on  $(0, \infty)$ :

$$\omega^{-\frac{1-1/\psi}{1-\gamma}} = \frac{\rho + \frac{1}{2}\gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta_H) - \left( 1 - \frac{1}{\psi} \right) \theta_H - \frac{1-1/\psi}{1-\gamma} \lambda_H (\omega^{-1} - 1)}{\rho + \frac{1}{2}\gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta_L) - \left( 1 - \frac{1}{\psi} \right) \theta_L - \frac{1-1/\psi}{1-\gamma} \lambda_L (\omega - 1)}. \quad (58)$$

In addition,  $\omega$  satisfies

$$\omega = \frac{H(\theta_H)}{H(\theta_L)}. \quad (59)$$

**Proof.** We first show that Eq. (58) has a unique solution on  $(\omega^*, 1)$ , where

$$\omega^* = 1 - \min \left\{ 1, \frac{\rho + \frac{1}{2}\gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta_L) - \left( 1 - \frac{1}{\psi} \right) \theta_L}{\frac{1-1/\psi}{\gamma-1} \lambda_L} \right\}. \quad (60)$$

Denote

$$LHS(\omega) = \omega^{-\frac{1-1/\psi}{1-\gamma}} \quad (61)$$

and

$$RHS(\omega) = \frac{\rho + \frac{1}{2}\gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta_H) - \left( 1 - \frac{1}{\psi} \right) \theta_H - \frac{1-1/\psi}{1-\gamma} \lambda_H (\omega^{-1} - 1)}{\rho + \frac{1}{2}\gamma \left( 1 - \frac{1}{\psi} \right) \sigma_C^2(\theta_L) - \left( 1 - \frac{1}{\psi} \right) \theta_L - \frac{1-1/\psi}{1-\gamma} \lambda_L (\omega - 1)}. \quad (62)$$

$LHS(\omega)$  is strictly increasing on  $(\omega^*, 1)$ ,  $LHS(0) = 0$ , and  $LHS(1) = 1$ . Also,  $RHS(\omega)$  is strictly decreasing on  $(\omega^*, 1)$ .  $RHS(\omega) \rightarrow +\infty$  as  $\omega \rightarrow 0$ , and  $RHS(1) < 1$ . This establishes the existence and uniqueness of the solution to Eq. (58) on  $(\omega^*, 1)$ . Using Eqs. (55)–(58), one can show that  $U_t$  in Eq. (55) satisfies the defining properties of the stochastic differential utility in the infinite horizon case, as given in Appendix C in Duffie and Epstein (1992b). ■

Using the results in Duffie and Epstein (1992b), the state price density of the economy is given by

$$\frac{d\pi_t}{\pi_t} = \frac{dF_C(C_t, V_t)}{F_C(C_t, V_t)} + F_V(C_t, V_t) dt. \quad (63)$$

Applying the generalized Ito's formula (Øksendal and Sulem, 2004), we can derive the expression of the pricing kernel and the risk-free interest rate of the economy. This is summarized in Lemma 2.

**Lemma 2.** *The state price density of the economy, denoted by  $\{\pi_t\}_{t \geq 0}$ , is a Levy process of the form*

$$d\pi_t = \pi_t \left[ -\bar{r}(\theta_t) dt - \gamma \sigma_C dB_t - \eta_\pi(\theta_t)^T dN_t \right]. \quad (64)$$

The risk-free interest rate,  $r(\theta)$ , is given by

$$\begin{aligned} r(\theta) &= \rho + \frac{1}{\psi} \theta - \frac{1}{2} \gamma \left( 1 + \frac{1}{\psi} \right) \sigma_C^2(\theta) + \lambda_H I_{\{\theta_H\}}(\theta) \left[ 1 - \hat{\omega}^{-1} + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} (\omega^{-1} - 1) \right] \\ &+ \lambda_L I_{\{\theta_L\}}(\theta) \left[ 1 - \hat{\omega} + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} (\omega - 1) \right]. \end{aligned} \quad (65)$$

The notations in the above equations are defined as

$$\eta_\pi(\theta) = [(1 - \hat{\omega}^{-1}) I_{\{\theta_H\}}(\theta), (1 - \hat{\omega}) I_{\{\theta_L\}}(\theta)], \quad (66)$$

$$\hat{\omega} = \omega^{\frac{1/\psi - \gamma}{1 - \gamma}}, \quad (67)$$

$$\bar{r}(\theta_H) = \rho + \frac{1}{\psi} \theta_H - \frac{1}{2} \gamma \left( 1 + \frac{1}{\psi} \right) \sigma_C^2(\theta_H) + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \lambda_H (\omega^{-1} - 1), \quad (68)$$

and

$$\bar{r}(\theta_L) = \rho + \frac{1}{\psi} \theta_L - \frac{1}{2} \gamma \left( 1 + \frac{1}{\psi} \right) \sigma_C^2(\theta_L) + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma} \lambda_L (\omega - 1). \quad (69)$$

For later reference, note that  $\hat{\omega} \in (0, 1)$  under our assumption of the preference parameters.

## A.2. Valuation of options and assets in place

In this section, we solve for the value of assets in place and growth options and provide a proof of Proposition 1.

### A.2.1. Valuation of assets in place

We first solve for the value function of assets in place. Using the generalized Ito's formula, the value function  $V_A(D, \theta)$  has to satisfy

$$\begin{aligned} &D - [\kappa + \bar{r}(\theta_H)] V_A(D, \theta_H) + [\mu_A + \theta_H - \gamma \sigma_C^2(\theta_H)] DV'(D, \theta_H) \\ &+ \frac{1}{2} [\sigma_A^2 + \sigma_C^2(\theta_H)] V''(D, \theta_H) + \lambda_H [\hat{\omega}^{-1} V(D, \theta_L) - V(D, \theta_H)] = 0 \end{aligned} \quad (70)$$

and

$$\begin{aligned} &D - [\kappa + \bar{r}(\theta_L)] V_A(D, \theta_L) + [\mu_A + \theta_L - \gamma \sigma_C^2(\theta_L)] DV'(D, \theta_L) \\ &+ \frac{1}{2} [\sigma_A^2 + \sigma_C^2(\theta_L)] V''(D, \theta_L) + \lambda_L [\hat{\omega} V(D, \theta_H) - V(D, \theta_L)] = 0. \end{aligned} \quad (71)$$

Because the value function of assets in place must be linear as in Eq. (20), the above two equations can be used to solve for  $a(\theta_H)$  and  $a(\theta_L)$ :

$$a(\theta_H) = \frac{\lambda_H \hat{\omega}^{-1} + \varrho_L + \kappa - \mu_A}{(\varrho_H + \kappa - \mu_A)(\varrho_L + \kappa - \mu_A) - \lambda_H \lambda_L}, \quad a(\theta_L) = \frac{\lambda_L \hat{\omega} + \varrho_H + \kappa - \mu_A}{(\varrho_H + \kappa - \mu_A)(\varrho_L + \kappa - \mu_A) - \lambda_H \lambda_L}, \quad (72)$$

where  $\varrho_H$  and  $\varrho_L$  are given by

$$\varrho_H = \rho - \left(1 - \frac{1}{\psi}\right) \theta_H + \frac{1}{2} \gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2(\theta_H) + \frac{1/\psi - \gamma}{1 - \gamma} \lambda_H (\omega^{-1} - 1) + \lambda_H, \quad (73)$$

and

$$\varrho_L = \rho - \left(1 - \frac{1}{\psi}\right) \theta_L + \frac{1}{2} \gamma \left(1 - \frac{1}{\psi}\right) \sigma_C^2(\theta_L) + \frac{1/\psi - \gamma}{1 - \gamma} \lambda_L (\omega - 1) + \lambda_L. \quad (74)$$

In the special case of  $\kappa = \mu_A = 0$ ,  $a(\theta)$  is the wealth-to-consumption ratio. In fact, let  $a_W(\theta_t)$  denote the wealth-to-consumption ratio. Then,

$$\frac{a_W(\theta_H)}{a_W(\theta_L)} = \frac{\lambda_H \hat{\omega}^{-1} + \varrho_L}{\lambda_L \hat{\omega} + \varrho_H} = \omega^{\frac{1-1/\psi}{1-\gamma}}, \quad (75)$$

where the first equality follows directly from Eq. (72) and the second follows from the definition of  $\omega$  in Eq. (58).

### A.2.2. Optimal stopping of growth options

Consider now the optimal stopping problem of growth options in Eq. (21), subject to the law of motion of the quality of options, Eq. (16), and the pricing kernel, Eq. (64).

**Lemma 3.** *The value function of growth options is of the form given in Eq. (27), where  $e_1 > 0$ , and  $e_2 < 0$  are the two solutions to the quadratic equation*

$$\lambda_L \hat{\omega} e^2 + (\varrho_H - \varrho_L) e - \lambda_H \hat{\omega}^{-1} = 0, \quad (76)$$

and  $\zeta_1, \zeta_2 > 0$  are the unique positive solutions to the quadratic equation

$$\frac{1}{2} \sigma_O^2 \zeta_i^2 + \left[ \mu_O - \frac{1}{2} \sigma_O^2 \right] \zeta_i - (\kappa + \varrho_L) + \lambda_L \hat{\omega} e_i = 0, \quad (77)$$



for  $i = 1, 2$ , respectively.

**Proof.** Using the generalized Ito's formula, the value function of the optimization problem in Eq. (21) must satisfy the coupled ordinary differential equations (ODEs)

$$\begin{aligned} -V_O(X, \theta_H) [\varrho_H + \kappa - \lambda_H] + \frac{\partial V_O}{\partial X}(X, \theta_H) X \mu_O + \frac{1}{2} \frac{\partial^2 V_O}{\partial^2 X}(X, \theta_H) X^2 \sigma_O^2 \\ + \lambda_H [\hat{\omega}^{-1} V_O(X, \theta_L) - V_O(X, \theta_H)] = 0 \end{aligned} \quad (78)$$

and

$$\begin{aligned} -V_O(X, \theta_L) [\varrho_L + \kappa - \lambda_L] + \frac{\partial V_O}{\partial X}(X, \theta_L) X \mu_O + \frac{1}{2} \frac{\partial^2 V_O}{\partial^2 X}(X, \theta_L) X^2 \sigma_O^2 \\ + \lambda_L [\hat{\omega} V_O(X, \theta_H) - V_O(X, \theta_L)] = 0. \end{aligned} \quad (79)$$

We seek a solution of the following form:

$$V_O(X, \theta_H) = eX^\zeta; \quad V_O(X, \theta_L) = X^\zeta. \quad (80)$$

It follows that Eqs. (78) and (79) can be rewritten as

$$-[\varrho_H + \kappa - \lambda_H] eX^\zeta + \mu_O e \zeta X^\zeta + \frac{1}{2} \sigma_O^2 e \zeta (\zeta - 1) X^\zeta + \lambda_H [\hat{\omega}^{-1} X^\zeta - eX^\zeta] = 0 \quad (81)$$

and

$$-[\varrho_L + \kappa - \lambda_L] X^\zeta + \mu_O \zeta X^\zeta + \frac{1}{2} \sigma_O^2 \zeta (\zeta - 1) X^\zeta + \lambda_L [\hat{\omega} e X^\zeta - X^\zeta] = 0, \quad (82)$$

respectively. Dividing both sides by  $X^\zeta$  and rearranging, we obtain

$$\frac{1}{2} \sigma_O^2 \zeta^2 + \left[ \mu_O - \frac{1}{2} \sigma_O^2 \right] \zeta - (\kappa + \varrho_H) + \lambda_H \hat{\omega}^{-1} e^{-1} = 0 \quad (83)$$

and

$$\frac{1}{2} \sigma_O^2 \zeta^2 + \left[ \mu_O - \frac{1}{2} \sigma_O^2 \right] \zeta - (\kappa + \varrho_L) + \lambda_L \hat{\omega} e = 0. \quad (84)$$

To obtain  $e$  and  $\zeta$ , let  $e$  be a solution to the quadratic equation

$$\lambda_H \hat{\omega}^{-1} e^{-1} - \varrho_H = \lambda_L \hat{\omega} e - \varrho_L, \quad (85)$$

then  $\xi$  can be found by solving Eq. (84). In particular, the two solutions to Eq. (85) are given by

$$e_1 = \frac{1}{2\lambda_L\hat{\omega}} \left\{ (\varrho_L - \varrho_H) + \sqrt{(\varrho_L - \varrho_H)^2 + 4\lambda_H\lambda_L} \right\} > 0, \quad (86)$$

and

$$e_2 = \frac{1}{2\lambda_L\hat{\omega}} \left\{ (\varrho_L - \varrho_H) - \sqrt{(\varrho_L - \varrho_H)^2 + 4\lambda_H\lambda_L} \right\} < 0. \quad (87)$$

For each of the two solutions, there are two solutions for  $\zeta$  to the quadratic Eq. (84), one of which is negative.

The boundary condition

$$\lim_{X \rightarrow 0} V_O(X, \theta) \geq 0, \quad \theta = \theta_H, \theta_L \quad (88)$$

rules out the possibility of negative  $\zeta$ . Let  $\zeta_1$  be the positive solution to Eq. (84) with  $e = e_1$ , and  $\zeta_2$  be the positive solution that corresponds to  $e = e_2$ . One can verify that  $\zeta_2 > \zeta_1 > 1$ . Finally, given the equilibrium price of capital goods,  $q(\theta_H)$  and  $q(\theta_L)$ , value-matching and smooth-pasting at the boundary determines the two option exercise thresholds,  $X^*(\theta_H)$ ,  $X^*(\theta_L)$ , and the constants,  $K_1$  and  $K_2$  in Eq. (27). This completes the proof. ■

### A.2.3. Optimality of nonstorage of capital goods

Let the equilibrium price of capital goods be  $q(\theta_H)$  and  $q(\theta_L)$ . Condition (22) implies that, for all  $s > t$ ,

$$e^{-\kappa_C t} \pi_t q(\theta_t) \geq E_t [\pi_s e^{-\kappa_C s} q(\theta_s)]. \quad (89)$$

This is equivalent to

$$\lim_{\Delta \rightarrow 0} \frac{1}{\Delta} E_t \left[ \pi_{t+\Delta} e^{-\kappa_C(t+\Delta)} q(\theta_{t+\Delta}) - e^{-\kappa_C t} \pi_t q(\theta_t) \right] \leq 0 \quad \text{for all } t. \quad (90)$$

Using Ito's formula and the law of motion of  $\pi_t$  in Eq. (64), the above is equivalent to

$$\kappa_C \geq -\bar{r}(\theta_H) + \lambda_H \left[ \hat{\omega}^{-1} \frac{q(\theta_L)}{q(\theta_H)} - 1 \right] \quad (91)$$

and

$$\kappa_C \geq -\bar{r}(\theta_L) + \lambda_L \left[ \hat{\omega} \frac{q(\theta_H)}{q(\theta_L)} - 1 \right], \quad (92)$$

where  $\bar{r}(\theta)$  is defined in Eqs. (68) and (69). Let

$$\kappa_C^* \equiv \max \left\{ -\bar{r}(\theta_H) + \lambda_H \left[ \hat{\omega}^{-1} \frac{q(\theta_L)}{q(\theta_H)} - 1 \right], -\bar{r}(\theta_L) + \lambda_L \left[ \hat{\omega} \frac{q(\theta_H)}{q(\theta_L)} - 1 \right], 0 \right\}. \quad (93)$$

Then, storage is never optimal in equilibrium as long as  $\kappa_C \geq \kappa_C^*$ .

### A.3. The cross-sectional distribution of blueprints and production units

Along the balanced growth path, the cross-sectional distribution of the quality of blueprints is time-invariant after being normalized by the common trend  $m_t$ . At any time  $t$ , the total measure of blueprints that enters into the economy is  $m_t$ . The total measure of blueprints being implemented is  $\delta m_t$  due to the market clearing condition and optimality of nonstorage. At the same time, fraction  $\kappa$  of existing options experiences a Poisson death shock because of the law of large numbers. Initially, at time 0, there are no blueprints in the economy, so the entry rate of blueprints,  $m(0)$ , is higher than the exit rate,  $\delta m(0)$ . Because there is more entry than exit, over time the total measure of blueprints increases. The economy converges to the balanced growth path as soon as the total measure of blueprints reaches  $\frac{1-\delta}{\kappa} m(t)$ , where the sum of the total measure of blueprints that leave the economy due to Poisson death shocks,  $\frac{1-\delta}{\kappa} m(t) \times \kappa = (1-\delta) m(t)$ , and those that exit due to option exercise,  $\delta m(t)$ , equals the entry rate. Below we show that not only the total measure of blueprints in the economy is time-invariant after being normalized by  $m(t)$ , the total measure of blueprints with any quality  $X$  is also time-invariant. That is, the cross-sectional distribution of the quality of blueprints is stationary after normalization.

We first set up some notations. Consider the following quadratic equation in  $\eta$

$$\kappa + \left( \mu_O - \frac{1}{2} \sigma_O^2 \right) \eta - \frac{1}{2} \sigma_O^2 \eta^2 = 0. \quad (94)$$

Denote the two roots of Eq. (94) as

$$\eta_1 = \left( \frac{\mu_O}{\sigma_O^2} - \frac{1}{2} \right) + \sqrt{\left( \frac{\mu_O}{\sigma_O^2} - \frac{1}{2} \right)^2 + \frac{2\kappa}{\sigma_O^2}} > 0 \quad (95)$$

and

$$\eta_2 = \left( \frac{\mu_O}{\sigma_O^2} - \frac{1}{2} \right) - \sqrt{\left( \frac{\mu_O}{\sigma_O^2} - \frac{1}{2} \right)^2 + \frac{2\kappa}{\sigma_O^2}} < 0. \quad (96)$$

We also frequently refer to the following two equations. The first is the forward equation for a family of

density functions indexed by the time variable  $l$ :

$$\nu_l(l, y) = -\kappa\nu(l, y) - \mu\nu_y(l, y) + \frac{1}{2}\sigma^2\nu_{yy}(l, y). \quad (97)$$

The second is an equation that defines the operator  $T$ . Given a family of density functions,  $\{\nu(l, \cdot)\}_{l>0}$ , indexed by time,  $T\nu$  is a density defined by

$$\forall y \in \mathbb{R}, \quad [T\nu](y) = \int_0^\infty \nu(l, y) dl. \quad (98)$$

Under some appropriate conditions to be discussed below,  $T\nu$  is the stationary distribution associated with  $\{\nu(l, \cdot)\}_{l>0}$ .

It is convenient to consider the distribution of log quality. We use lower cases to denote logs:

$$x = \ln X, \quad x^* = \ln X^*, \quad x_0 = \ln X_0. \quad (99)$$

Consider a blueprint with quality  $x_s = x$  at time  $s$ . For  $l > 0$ , let  $\nu_O(l, \cdot | x) \frac{m_{s+l}}{m_s}$  be the density of the log quality of the blueprint at time  $s+l$ , that is, for any interval  $(y_1, y_2)$ ,  $\int_{y_1}^{y_2} \nu_O(l, y | x) dy \times \frac{m_{s+l}}{m_s}$  is the probability of the event  $x_{s+l} \in (y_1, y_2)$ . We first show that, for  $\forall l > 0$  and  $\forall y \neq x$ ,  $\nu_O(l, y | x)$  has to satisfy the forward Eq. (97) with  $\mu = \mu_O - \frac{1}{2}\sigma_O^2$  and  $\sigma = \sigma_O$ . In addition,  $\nu_O(l, y | x)$  satisfies the boundary conditions

$$\forall l > 0, \quad \nu_O(l, x^* | x) = 0; \quad \lim_{y \rightarrow -\infty} \nu_O(l, y | x) = 0 \quad (100)$$

and

$$\forall y \neq x, \quad \lim_{l \rightarrow 0} \nu_O(l, y | x) = 0. \quad (101)$$

The above claim is formalized in Lemma 4.

**Lemma 4.**  $\nu_O(l, y | x)$  satisfies conditions (97), (100), and (101).

**Proof.** By Ito's formula, the law of motion of log quality  $x_{s+l}$  is given by

$$dx_{s+l} = \left( \mu_O - \frac{1}{2}\sigma_O^2 \right) dl + \sigma_O dB_{s+l}^i. \quad (102)$$

We use the following locally consistent Markov chain approximation of the diffusion process in Eq. (102)

(Kushner and Dupuis, 2001). For an infinitesimal  $h$ , let

$$\Delta = \frac{h^2}{\mu_O h + \sigma_O^2}. \quad (103)$$

The transition probability of the Markov chain is

$$\Pr(y, y+h) = \frac{\mu_O h + \frac{1}{2}\sigma_O^2}{\mu_O h + \sigma_O^2}; \quad \Pr(y, y-h) = \frac{\frac{1}{2}\sigma_O^2}{\mu_O h + \sigma_O^2}. \quad (104)$$

Suppose, for  $\forall y$ , the total measure of blueprints at location  $y$  at time  $s+l$  is  $\nu_O(l, y) \frac{m_{s+l}}{m_s}$ . At time  $s+l+\Delta$ , the total measure at location  $y$  comes from two sources: the mass at location  $y-h$  at time  $s+l$  and the mass at location  $y+h$  at time  $s+l$ . The total measure of blueprints at location  $y-h$  at time  $s+l$  is  $\nu_O(l, y-h) \frac{m_{s+l}}{m_s}$ . It increases by  $e^{-\kappa\Delta} \times \frac{m_{s+l+\Delta}}{m_{s+l}}$  during the time interval  $[s+l, s+l+\Delta]$ , as blueprints are used to produce new blueprints with identical quality. These blueprints visit location  $y$  with probability  $\Pr(y-h, y)$ . Therefore, the total measure of blueprints that come from this source is

$$\nu_O(l, y-h) \times \frac{m_{s+l}}{m_s} \times e^{-\kappa\Delta} \times \frac{m_{s+l+\Delta}}{m_{s+l}} \times \Pr(y-h, y) = \nu_O(l, y-h) \times \frac{m_{s+l+\Delta}}{m_s} \times e^{-\kappa\Delta} \times \Pr(y-h, y). \quad (105)$$

By the same logic, the total measure of blueprints that come from the mass at location  $y+h$  at time  $s+l$  is

$$e^{-\kappa\Delta} \times \nu_O(l, y+h) \frac{m_{s+l+\Delta}}{m_s} \Pr(y+h, y). \quad (106)$$

Therefore, the total measure of blueprints at location  $y$  at time  $s+l$  is

$$\begin{aligned} \nu_O(l+\Delta, y) \frac{m_{s+l+\Delta}}{m_s} &= e^{-\kappa\Delta} \times \nu_O(l, y-h) \times \frac{m_{s+l+\Delta}}{m_s} \Pr(y-h, y) \\ &\quad + e^{-\kappa\Delta} \times \nu_O(l, y+h) \frac{m_{s+l+\Delta}}{m_s} \Pr(y+h, y). \end{aligned} \quad (107)$$

Eq. (107) can be written as

$$\nu_O(l+\Delta, y) = e^{-\kappa\Delta} [\nu_O(l, y-h) \Pr(y-h, y) + \nu_O(l, y+h) \Pr(y+h, y)]. \quad (108)$$

Subtracting  $\nu_O(l, y|x)$ , dividing by  $\Delta$ , and taking the limit as  $\Delta \rightarrow 0$  of both sides of Eq. (108), we obtain Eq. (97). ■

The solution to the partial differential equation (97) with the boundary conditions, Eqs. (100) and (101),

is given by

$$\begin{aligned}\nu(l, y) &= e^{-\kappa a} n(t\sigma^2, y - x - \mu a) \\ &\quad - e^{-\frac{2\mu(x-\bar{x})}{\sigma^2}} n(t\sigma^2, y + x - \mu a - 2\bar{x}),\end{aligned}\tag{109}$$

where

$$n(t, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}\tag{110}$$

and  $\mu = \mu_O - \frac{1}{2}\sigma_O^2$ ,  $\sigma = \sigma_O$ , and  $\bar{x} = x^*$ . This solution can be found in Luttmer (2007).

Suppose the density of the log quality of blueprints at time 0 is  $\phi(x) m_0$ , then the density of log quality at time  $t$  is given by

$$\begin{aligned}&\int_{-\infty}^{+\infty} \nu(t, y|x) \frac{m_t}{m_0} \times \phi(x) m_0 dx + \int \nu(t-u, y|x_0) \frac{m_t}{m_u} \times m_u du \\ &= m_t \left[ \int_{-\infty}^{+\infty} \nu(t, y|x) \phi(x) dx + \int \nu(t-u, y|x_0) du \right],\end{aligned}\tag{111}$$

where the first term is the density of blueprints existing at time 0, and the second term is the integral of the density of all blueprints that arrive during the time interval  $(0, t)$ . We define a stationary density  $\phi_O(y|x_0) = T\nu_O(y|x_0)$ , where the operator  $T$  is given in Eq. (98). Then, along the balance growth path, the density of the log quality of blueprints is given by  $\phi_O(y|x_0) m_t$ . This claim is formalized by Lemma 5.

**Lemma 5.** *Suppose  $\nu(l, y|x)$  satisfies the boundary conditions (100) and (101). Suppose also that  $\kappa > 0$ , then the integral in Eq. (98) exists, and  $T\nu(y|x)$  is given by*

$$T\nu(y|x) = \begin{cases} \frac{1}{\sqrt{\mu^2 + 2\kappa\sigma^2}} e^{-\eta_2(x-\bar{x})} [e^{\eta_2(y-\bar{x})} - e^{\eta_1(y-\bar{x})}] & \text{if } y \geq x \\ \frac{1}{\sqrt{\mu^2 + 2\kappa\sigma^2}} [e^{-\eta_1(x-\bar{x})} - e^{\eta_2(x-\bar{x})}] e^{\eta_1(y-\bar{x})} & \text{if } y < x \end{cases}.\tag{112}$$

Also,  $T\nu(y|x)$  satisfies

$$T\nu(y|x) = \int_{-\infty}^{+\infty} \nu(t, y|x') T\nu(x'|x) dx + \int \nu(t-u, y|x) du.\tag{113}$$

In the above equations,  $x = x_0$ ,  $\bar{x} = x^*$ ,  $\mu = \mu_O - \frac{1}{2}\sigma_O^2$ ,  $\sigma = \sigma_O$ , and  $\eta_1$  and  $\eta_2$  are defined in Eqs. (95) and (96).

**Proof of Proposition 2.** Let  $\phi_O(y|x_0) = T\nu_O(y|x_0)$  be the stationary density of the log quality of

blueprints, and let  $\Phi$  be the corresponding stationary density of the quality of blueprints. The absorbing rate of  $\Phi$  at the absorbing barrier  $X^*$  can be calculated as

$$m_{EXIT}[\Phi, X^*] = \frac{1}{2}\sigma_O^2 |\phi'(x^*)| = \left(\frac{X_0}{X^*}\right)^{-\eta_2}. \quad (114)$$

We can then solve Eq. (23) and show that market clearing  $X^*$  is given by Eq. (30).

**Proof of Conjecture 1.** Consider a production unit with log dividend level  $\ln D_s^i = x$  at time  $s$ . Let  $\nu_A(l, y|x)$  be the density of  $\ln\left(D_{s+l}^i \frac{m_s}{m_{s+l}}\right)$ . Using the same argument as in the proof of Lemma 4, we can show that  $\nu_A(l, y|x)$  is given by Eq. (109) with  $\mu = \mu_O - \frac{1}{2}\sigma_O^2$ ,  $\sigma = \sigma_O$ , and  $\bar{x} = \infty$ . Define  $\phi_A(y) = T\nu_A(y|x^*)$  and

$$D_0 = \delta \times m_0 \times \int_{-\infty}^{\infty} e^y \phi_A(y) dy. \quad (115)$$

Suppose, at time 0, the economy is endowed with measure  $D_0$  of an initial generation of production units with identical dividend level of 1. Suppose also that the law of motion of the dividend paid by production units follows dynamics given in Eq. (18). Then, at time  $t$ , the total dividend paid by the initial generation of production units is

$$\begin{aligned} & D_0 \times \int_{-\infty}^{+\infty} e^y \frac{m_t}{m_0} \nu(t, y|0) dy \\ = & D_0 \times \int_{-\infty}^{+\infty} e^{y-x} \frac{m_t}{m_0} \nu(t, y|x) dy \\ = & \delta \times m_0 \times \int_{-\infty}^{\infty} e^x \phi_A(x) dx \times \int_{-\infty}^{+\infty} e^{y-x} \frac{m_t}{m_0} \nu(t, y|x) dy \\ = & \delta \times m_t \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^y \nu(t, y|x) \phi_A(x) dy dx, \end{aligned} \quad (116)$$

where the first equality uses a change of variables:

$$\forall x, \int_{-\infty}^{+\infty} e^y \nu(t, y|0) dy = \int_{-\infty}^{+\infty} e^{y-x} \nu(t, y|x) dy, \quad (117)$$

and the rest of the argument follows from Fubini's theorem.

For any  $u \in (0, t)$ , production units enter the economy with initial log dividend level  $x^*$  at rate  $\delta m_u$ . Therefore, at time  $t$ , the total dividend produced by generation- $u$  production units is

$$\delta m_u \times \int_{-\infty}^{\infty} e^y \frac{m_t}{m_u} \nu(t-u, y|x^*) dy = \delta m_t \times \int_{-\infty}^{\infty} e^y \nu(t-u, y|x^*) dy. \quad (118)$$

Thus, the amount of consumption goods at time  $t$  produced by production units that arrive during the interval  $(0, t)$  is given by

$$\delta m_t \times \int_0^t \int_{-\infty}^{\infty} e^y \nu(t-u, y | x^*) dy du. \quad (119)$$

Using Eqs. (116) and (119), the total amount of consumption goods produced at time  $t$  is, therefore,

$$\begin{aligned} & \delta \times m_t \times \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^y \nu(t, y | x) \phi_A(x) dy dx + \int_0^t \int_{-\infty}^{\infty} e^y \nu(t-u, y | x^*) dy du \right\} \\ = & \delta \times m_t \times \left\{ \int_{-\infty}^{\infty} e^y \left[ \int_{-\infty}^{\infty} \nu(t, y | x) \phi_A(x) dx + \int_0^t \nu(t-u, y | x^*) du \right] dy \right\} \\ = & \delta \times m_t \times \left\{ \int_{-\infty}^{\infty} e^y \phi_A(y) dy \right\} \\ = & D_0 \times \frac{m_t}{m_0}, \end{aligned} \quad (120)$$

which follows from the property in Eq. (113) applied to  $T\nu_A(y | x)$ . This proves the conjecture. The above analysis relies on the assumption that the integral in Eq. (115) is finite. Lemma 6 provides conditions for the finiteness of the integral.

**Lemma 6.** *Suppose  $\mu_A - \kappa < 0$ , then the integral in Eq. (115) is finite. Furthermore, if we assume that  $\exists x_{MAX}$  such that  $x^* < x_{MAX} < \infty$  and, for all production units of generation  $s$ ,  $\forall s > 0$ , the following condition holds:*

$$\forall t > 0, \quad \ln \left( D_{s+t}^i \frac{m_s}{m_{s+t}} \right) \leq x_{MAX}, \quad (121)$$

then the integral in Eq. (115) is always finite.<sup>17</sup>

#### A.4. Asset pricing implications

**Proof of Proposition 3.** Using the expression of the state price density in Eq. (64), and the functional form of the value function of growth options and assets in place, one can solve for the risk premium of growth options and assets in place by applying the Ito's formula.

**Proof of Proposition 4.** Here we show that if the parameters of the model satisfy the following condition:

$$\frac{\lambda_H \hat{\omega}^{-1} + \varrho_L + \kappa - \mu_A}{\lambda_L \hat{\omega} + \varrho_H + \kappa - \mu_A} > \frac{1}{2\lambda_L \hat{\omega}} \left\{ (\varrho_L - \varrho_H) + \sqrt{(\varrho_L - \varrho_H)^2 + 4\lambda_H \lambda_L} \right\}, \quad (122)$$

<sup>17</sup>As shown in the lemma, if the condition  $\mu_A - \kappa < 0$  is not satisfied, we can impose an upper bound on the de-trended dividend process so that the integral Eq. (115) is always finite. This modification affects the valuation of assets in this economy. However, as long as we choose  $x_{MAX}$  to be large enough, the asset pricing implications of the model go through without change. The extension of the model to include this case is available upon request.



then the conclusion of the proposition holds.

By the definition of  $a(\theta)$  and  $e_1$ , condition (122) is equivalent to  $\frac{a(\theta_H)}{a(\theta_L)} > e_1$ . We first prove that the above condition implies  $K_2 < 0$ , where  $K_2$  is the constant in the value function of growth options in Eq. (27). Given the functional form of  $V_O(X, \theta)$  in Eq. (27),  $K_2$  is determined by the two smooth-pasting conditions in Eq. (29). In particular,

$$K_2 = \frac{1}{\zeta_2} (X^*)^{1-\zeta_2} \frac{e_1 a(\theta_L) - a(\theta_H)}{e_1 - e_2}. \quad (123)$$

By Lemma 3,  $e_1 - e_2 > 0$ . Therefore, condition (122) implies that  $K_2 < 0$ .

Now consider an option with quality  $0 < X \leq X^*$ . It follows that

$$\frac{V_O(X, \theta_H)}{V_O(X, \theta_L)} = \frac{K_1 e_1 X^{\zeta_1} + K_2 e_2 X^{\zeta_2}}{K_1 X^{\zeta_1} + K_2 X^{\zeta_2}} < \frac{K_1 e_1 X^{\zeta_1}}{K_1 X^{\zeta_1}} = e_1 < \frac{a(\theta_H)}{a(\theta_L)}, \quad (124)$$

where the first inequality is due to  $e_2 < 0$ , and the second inequality follows from condition (122). This completes the proof. ■

Finally, under the assumptions on the technology and preference parameters, the left-hand side of condition (122) is increasing in  $\mu_A$  and unbounded from above. Therefore, we can choose  $\bar{\mu}$  to be the greatest lower bound of  $\mu_A$  such that the condition holds.

**Proof of Proposition 5.** Consider a continuous time limit of  $\alpha_{t,\Delta}^i$  and  $\beta_{t,\Delta}^i$ , defined as

$$\alpha_t^i = \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} \alpha_{t,\Delta}^i, \quad \beta_t^i = \lim_{\Delta \rightarrow 0} \beta_{t,\Delta}^i. \quad (125)$$

By Eqs. (42) and (43), the theoretical values of  $\alpha_t^i$  is given by the following equations:

$$\alpha_t^i = (\gamma - \chi(\theta_H)) \sigma_C^2(\theta_H) + \lambda_H \left[ (1 - \hat{\omega}^{-1}) - \chi(\theta_H) \left( \frac{a_W(\theta_L)}{a_W(\theta_H)} - 1 \right) \right] \left[ \frac{V(i, t, \theta_L)}{V(i, t, \theta_H)} - 1 \right] \quad \text{if } \theta_t = \theta_H \quad (126)$$

and

$$\alpha_t^i = (\gamma - \chi(\theta_L)) \sigma_C^2(\theta_L) + \lambda_L \left[ (1 - \hat{\omega}) - \chi(\theta_L) \left( \frac{a_W(\theta_H)}{a_W(\theta_L)} - 1 \right) \right] \left[ \frac{V(i, t, \theta_H)}{V(i, t, \theta_L)} - 1 \right] \quad \text{if } \theta_t = \theta_L, \quad (127)$$

where

$$\chi(\theta_t) = \lim_{\Delta \rightarrow 0} \frac{E_t [R_{t,t+\Delta}^W] - r_{t,t+\Delta}}{\text{Var}_t (R_{t,t+\Delta}^W)}. \quad (128)$$

To prove that assets with higher exposure to long-run risks obtain higher alphas, we need to show that

$$(\hat{\omega}^{-1} - 1) > \chi(\theta_H) \left( 1 - \frac{a_W(\theta_L)}{a_W(\theta_H)} \right) \quad \text{and} \quad (1 - \hat{\omega}) > \chi(\theta_L) \left( \frac{a_W(\theta_H)}{a_W(\theta_L)} - 1 \right). \quad (129)$$

We prove only the first inequality in Eq. (129). The second inequality can be established by the same argument. Taking continuous time limit of Eq. (128) and using Eq. (75), we can show that

$$\chi(\theta_H) = \frac{\gamma\sigma_C^2(\theta_H) + \lambda_H(\hat{\omega}^{-1} - 1)\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right)}{\sigma_C^2(\theta_H) + \lambda_H\left(\omega^{-\frac{1-1/\psi}{1-\gamma}} - 1\right)^2}. \quad (130)$$

Therefore, the right-hand side of the first inequality in Eq. (129) can be written as

$$\chi(\theta_H)\left(1 - \frac{a_W(\theta_L)}{a_W(\theta_H)}\right) = \frac{\gamma\sigma_C^2(\theta_H)\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right) + \lambda_H(\hat{\omega}^{-1} - 1)\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right)^2}{\sigma_C^2(\theta_H) + \lambda_H\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right)^2}. \quad (131)$$

This implies that

$$\min\left\{\gamma\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right), \hat{\omega}^{-1} - 1\right\} \leq \chi(\theta_H)\left(1 - \frac{a_W(\theta_L)}{a_W(\theta_H)}\right) \leq \max\left\{\gamma\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right), \hat{\omega}^{-1} - 1\right\}. \quad (132)$$

To establish the first inequality in Eq. (129), it is enough to show that

$$(\hat{\omega}^{-1} - 1) > \gamma\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right). \quad (133)$$

By Eq. (67), the above condition is equivalent to

$$\omega^{\frac{\gamma-1/\psi}{1-\gamma}} - 1 - \gamma\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right) > 0. \quad (134)$$

Eq. (134) is always true because  $\omega < 1$  (the function  $\omega^{\frac{\gamma-1/\psi}{1-\gamma}} - 1 - \gamma\left(1 - \omega^{-\frac{1-1/\psi}{1-\gamma}}\right)$  is decreasing on  $(0, 1)$  and reaches 0 at 1 under the parameter restriction  $\gamma > \psi > 1$ ). This completes the proof. ■

#### A.5. Cash flow duration of options and assets in place

In this section, we derive the Macaulay duration for options and assets in place as defined in Eq. (44). We consider a more general dividend process described in Eq. (48). We first solve for the value of assets in place and growth options under these generalized dynamics. Using a similar argument as in Subsection A.2.1, one can show that the value of assets in place is linear as in Eq. (20). The coefficients  $a(\theta_H)$  and  $a(\theta_L)$  are of the same form as in Eq. (72), with the following alternative definitions of  $\varrho_H$  and  $\varrho_L$ :

$$\varrho_H = \rho + \frac{1}{\psi}\theta_H + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2(\theta_H) + \frac{1/\psi - \gamma}{1-\gamma}\lambda_H(\omega^{-1} - 1) + \lambda_H - \mu_A(\theta_H) + \mu_A \quad (135)$$

and

$$\varrho_L = \rho + \frac{1}{\psi}\theta_L + \frac{1}{2}\gamma\left(1 - \frac{1}{\psi}\right)\sigma_C^2(\theta_L) + \frac{1/\psi - \gamma}{1 - \gamma}\lambda_L(\omega - 1) + \lambda_L - \mu_A(\theta_L) + \mu_A. \quad (136)$$

Applying the argument of Subsection A.2.2, one can show that the value of growth options in this generalized case satisfies the coupled ODE in Eqs. (78) and (79) and is characterized by Lemma 3 with  $\varrho_H$  and  $\varrho_L$ . In the special case of  $\mu_A(\theta) = \mu_A$ , for all  $\theta$ , we recover the formula for assets in place and growth options in the general equilibrium model.

By definition, the Macaulay duration of assets in place at time  $t$  is given by

$$\frac{1}{V_A(\theta_t, D_t)} E_t \left[ \int_t^\infty (s-t) \times \frac{\pi_s}{\pi_t} e^{-\kappa(s-t)} D_s ds \right] = \frac{1}{a(\theta_t)} E_t \left[ \int_t^\infty (s-t) \times \frac{\pi_s}{\pi_t} e^{-\kappa(s-t)} \frac{D_s}{D_t} ds \right]. \quad (137)$$

Given the law of motion of dividend and the state price density, it follows that duration of assets in place depends only on  $\theta_t$ . Denoting duration of assets in place by  $M_A(\theta_t)$ , the above equation implies

$$\begin{aligned} a(\theta_t) M_A(\theta_t) D_t &= E_t \left[ \int_t^\infty (s-t) \times \frac{\pi_s}{\pi_t} e^{-\kappa(s-t)} D_s ds \right] \\ &= E_t \left[ \int_t^\infty s \times \frac{\pi_s}{\pi_t} e^{-\kappa(s-t)} D_s ds \right] - t a(\theta_t) D_t \\ &= \frac{e^{\kappa t}}{\pi_t} E_t \left[ \int_t^\infty s \times \pi_s e^{-\kappa s} D_s ds \right] - t a(\theta_t) D_t. \end{aligned} \quad (138)$$

Multiplying both sides by  $e^{-\kappa t} \pi_t$  and rearranging the equation, we obtain

$$e^{-\kappa t} \pi_t a(\theta_t) D_t [M_A(\theta_t) + t] + \int_0^t s \times \pi_s e^{-\kappa s} D_s ds = E_t \left[ \int_0^\infty s \times \pi_s e^{-\kappa s} D_s ds \right]. \quad (139)$$

The right-hand side of the above equation is a martingale. Using Ito's lemma, one can show that  $M(\theta)$  must satisfy

$$M_A(\theta_H) = \frac{1}{\varrho_H + \kappa - \mu_A - \lambda_H \hat{\omega}^{-1} \zeta^{-1}} \quad (140)$$

and

$$M_A(\theta_L) = \frac{1}{\varrho_L + \kappa - \mu_A - \lambda_L \hat{\omega} \zeta}, \quad (141)$$

where  $\varrho_H$  and  $\varrho_L$  are defined in Eqs. (135) and (136) and

$$\varsigma = \frac{\frac{a(\theta_H)}{a(\theta_L)} (\varrho_L + \kappa - \mu_A) + \lambda_H \hat{\omega}^{-1}}{\varrho_H + \kappa - \mu_A + \frac{a(\theta_H)}{a(\theta_L)} \lambda_L \hat{\omega}}. \quad (142)$$

In the above equation,  $a(\theta_H)$  and  $a(\theta_L)$  are of the same form as in Eq. (72) based on the definition of  $\varrho_H$  and  $\varrho_L$  in Eqs. (135) and (136).

Next, consider duration of growth options. By definition, the Macaulay duration of growth options at time  $t$  is given by

$$M_O(\theta_t, X_t) = \frac{1}{V_O(\theta_t, X_t)} \left\{ E_t \left[ \frac{m_\tau}{m_t} \int_\tau^\infty (s-t) e^{-\kappa(s-t)} \frac{\pi_s}{\pi_t} D_s ds \right] - E_t \left[ e^{-\kappa(\tau-t)} \frac{m_\tau}{m_t} (\tau-t) \frac{\pi_\tau}{\pi_t} q_\tau ds \right] \right\}, \quad (143)$$

where  $\tau$  is the optimal stopping time for option exercise. The first term in the bracket can be written as

$$\begin{aligned} & E_t \left[ \frac{m_\tau}{m_t} \int_\tau^\infty (s-t) e^{-\kappa(s-t)} \frac{\pi_s}{\pi_t} D_s ds \right] \\ &= E_t \left[ e^{-\kappa(\tau-t)} \frac{m_\tau}{m_t} \frac{\pi_\tau}{\pi_t} \left( E_\tau \left\{ \int_\tau^\infty (s-\tau) e^{-\kappa(s-\tau)} \frac{\pi_s}{\pi_\tau} D_s ds \right\} + (\tau-t) E_\tau \left\{ \int_\tau^\infty e^{-\kappa(s-\tau)} \frac{\pi_s}{\pi_\tau} D_s ds \right\} \right) \right] \\ &= E_t \left[ e^{-\kappa(\tau-t)} \frac{m_\tau}{m_t} \{ M_A(\theta_\tau) a(\theta_\tau) D_\tau + (\tau-t) a(\theta_\tau) D_\tau \} \right], \end{aligned} \quad (144)$$

where the last equality uses the fact that  $V_A(\theta, D) = a(\theta) D$ . Therefore,

$$\begin{aligned} V_O(\theta_t, X_t) M_O(\theta_t, X_t) &= E_t \left[ e^{-\kappa(\tau-t)} \frac{m_\tau}{m_t} \frac{\pi_\tau}{\pi_t} \{ M_A(\theta_\tau) a(\theta_\tau) D_\tau + (\tau-t) [a(\theta_\tau) D_\tau - q(\theta_\tau)] \} \right] \\ &= E_t \left[ e^{-\kappa(\tau-t)} \frac{m_\tau}{m_t} \frac{\pi_\tau}{\pi_t} \{ M_A(\theta_\tau) a(\theta_\tau) D_\tau + \tau [a(\theta_\tau) D_\tau - q(\theta_\tau)] \} \right] \\ &\quad - t E_t \left[ e^{-\kappa(\tau-t)} \frac{m_\tau}{m_t} \frac{\pi_\tau}{\pi_t} [a(\theta_\tau) D_\tau - q(\theta_\tau)] \right]. \end{aligned} \quad (145)$$

This implies that

$$\begin{aligned} & e^{-\kappa t} \pi_t m_t V_O(\theta_t, X_t) M_O(\theta_t, X_t) \\ &= E_t \left[ e^{-\kappa \tau} m_\tau \pi_\tau \{ M_A(\theta_\tau) a(\theta_\tau) D_\tau + \tau [a(\theta_\tau) D_\tau - q(\theta_\tau)] \} \right] - t E_t \left[ e^{-\kappa \tau} m_\tau \pi_\tau [a(\theta_\tau) D_\tau - q(\theta_\tau)] \right]. \end{aligned} \quad (146)$$

Let  $Q(\theta, X) = V_O(\theta, X) M_O(\theta, X)$ . Because  $E_t \left[ e^{-\kappa \tau} m_\tau \pi_\tau \{ M_A(\theta_\tau) a(\theta_\tau) D_\tau + \tau [a(\theta_\tau) D_\tau - q(\theta_\tau)] \} \right]$  and

$E_t [e^{-\kappa\tau} m_\tau \pi_\tau [a(\theta_\tau) D_\tau - q(\theta_\tau)]]$  are martingales, we have

$$\begin{aligned} & \lim_{\Delta \rightarrow 0} \frac{1}{\Delta} E_t [e^{-\kappa t + \Delta} \pi_{t+\Delta} m_{t+\Delta} Q(\theta_{t+\Delta}, X_{t+\Delta}) - e^{-\kappa t} \pi_t m_t Q(\theta_t, X_t)] \\ &= -E_t [e^{-\kappa\tau} m_\tau \pi_\tau [a(\theta_\tau) D_\tau - q(\theta_\tau)]] = e^{-\kappa t} \pi_t m_t V_O(\theta_t, X_t). \end{aligned} \quad (147)$$

Using Ito's formula, and the fact that  $V_O(\theta, X)$  satisfies the coupled ODEs in Eqs. (78) and (79), one can show that  $Q(\theta, X)$  must satisfy the same ODEs with  $\varrho_H$  and  $\varrho_L$  defined in Eqs. (135) and (136). This implies that  $Q(\theta, X)$  must be of the form in Eq. (47), where  $e_1$  and  $e_2$  are given in Eqs. (86) and (87) with the new definition of  $\varrho_H$  and  $\varrho_L$ . Finally, the constant  $L_1$  and  $L_2$  are determined by the value-matching condition implied by Eq. (147):

$$Q(\theta, X^*) = M_A(\theta) a(\theta) X^*, \quad \text{for } \theta = \theta_H, \theta_L. \quad (148)$$

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