Lecture 2: Optimal Portfolio Choice

I Optimal Portfolio Choice with One Risky Asset

Suppose an expected utility maximizing consumer has total wealth w. He can choose to invest in a risky asset with rate of return R (a random variable), and a risk-free asset with deterministic rate of return R_f. If the consumer chooses to invest α in the risky asset, and the remaining $w - \alpha$ of his wealth in the risk-free asset, his consumption is $c = wR_f + \alpha(R - R_f)$, a random variable. The consumer faces the following problem:

$$\max_{\alpha \in (-\infty,\infty)} E[u(wR_f + \alpha(R - R_f))]$$

Claim 1 *Suppose $u' > 0, u'' < 0$. Suppose also the optimal portfolio choice α^* exists. Then we have*

$$\begin{cases} \alpha^* > 0 \text{ if } E[R] - R_f > 0 \\ \alpha^* = 0 \text{ if } E[R] - R_f = 0 \\ \alpha^* < 0 \text{ if } E[R] - R_f < 0 \end{cases}$$

Proof. Here we only prove the first part of the claim, $\alpha^* > 0$ if $E[R] - R_f > 0$.

We assume without prove here that $E[u(wR_f + \alpha(R - R_f))]$, viewed as a function of α is of class C^2. Since the optimum exist, α^* has to satisfy:

$$E[(u'(wR_f + \alpha(R - R_f))(R - R_f)] = 0 \quad at \quad \alpha^*.$$

We denote

$$f(\alpha) = E[(u'(wR_f + \alpha(R - R_f))(R - R_f)].$$

Note $f(\alpha)$ must be a strictly decreasing function of α. To see this,

$$f'(\alpha) = E[(u''(wR_f + \alpha(R - R_f))(R - R_f)^2] < 0$$

as $u'' < 0$.

Evaluating $f(\alpha)$ at $\alpha = 0$, we have

$$\begin{aligned} f(0) &= E[(u'(wR_f)(R - R_f)] \\ &= u'(wR_f)(E[R] - R_f) > 0 \text{ since } E[R] - R_f > 0 \text{ and } u' > 0 \end{aligned}$$

$f(\alpha) > 0$ at $\alpha = 0$. In addition, $f(\alpha)$ is strictly decreasing, therefore, $f(\alpha) = 0$ must occur at $\alpha^* > 0$. ■

Exercise 1 *Complete the proof for the cases when* $E[R] - R_f = 0$ *and* $E[R] - R_f < 0$.

Remark 1 *Intuition: Consider the following Taylor expansion :* $u(x) = u'(\overline{x})(x - \overline{x}) + \frac{1}{2}u''(\overline{x})(x - \overline{x})^2 + o\|x - \bar{x}\|$. *We have:*

$$u(wR_f + \alpha(R - R_f)) = u(w\bar{r}) + u'(w\bar{r}) \times \alpha(R - R_f) + \frac{1}{2}u''(w\bar{r}) \times \alpha^2(R - R_f)^2 + o\|R - R_f\|,$$

Hence:

$$\begin{aligned} E[u(wR_f + \alpha(R - R_f))] &= u(w\bar{r}) + u'(w\bar{r}) \times \alpha E[(R - R_f)] + \frac{1}{2}u''(w\bar{r}) \times \alpha^2 E\left[(R - R_f)^2\right] \\ &= u(w\bar{r}) + u'(w\bar{r}) \times \alpha(\mu - R_f) + \frac{1}{2}u''(w\bar{r}) \times \alpha^2\left(\mu^2 + \sigma^2\right) \end{aligned}$$

Note if

$$\mu - R_f > 0, \quad \left(\mu^2 + \sigma^2\right) > 0, \quad u' > 0, \quad u'' < 0.$$

Therefore, around $\alpha = 0$, *an increase in* α *will always increase utility, because monotonicity is the property of the first order derivative, and risk aversion is the property of the second order derivative of the utility function. In this sense, expected utility displays second order risk aversion, that is risk aversion a second order concern when compared to monotonicity.*

Claim 2 *Suppose* $E[R] - R_f$ *is small. Then* $\alpha = \frac{1}{A(wR_f)\sigma^2}(\mu - R_f) + o\|\mu - R_f\|$, *where* $\mu = E[R]$ *and* $A(x) = -\frac{u''(x)}{u'(x)}$ *is the Arrow-Pratt measure of absolute risk aversion.*

Proof. Again, we assume that $E[u(wR_f + \alpha(R - R_f))]$, viewed as a function of α is of class C^2. Given that the optimal solution exist,

$$E[(u'(wR_f + \alpha(R - R_f))(R - R_f)] = 0 \text{ at } \alpha = \alpha^*.$$

We can write $R = \mu + \varepsilon$, where $\varepsilon = R - \mu$ is the deviation of return R from mean. The above can then be written as:

$$E[(u'(wR_f + \alpha(\mu + \varepsilon - R_f))(\mu + \varepsilon - R_f)] = 0.$$

The above FOC defines α^* implicitly as a function of $\mu = E[R]$, $\alpha(\mu)$. Consider a first order Taylor approximation of $\alpha(\mu)$ around R_f (the implicit function theorem says that

$\alpha(\mu)$ is continuously differentiable under certain conditions):

$$\alpha(\mu) = \alpha(R_f) + \alpha'(R_f)(\mu - R_f) + o\|\mu - R_f\|. \tag{1}$$

We know that $\alpha(R_f) = 0$ (see claim 1), therefore $\alpha(\mu) = \alpha'(R_f)(\mu - R_f) + o\|\mu - R_f\|$.

Now, we want to evaluate $\alpha'(R_f)$.Differentiate the FOC with respect to μ, and denote $c^* = wr + \alpha(\mu + \varepsilon - r)$:

$$E[u''(c^*)\{\alpha'(\mu)(\mu + \varepsilon - R_f) + \alpha(\mu)\}(\mu + \varepsilon - R_f) + u'(c^*)] = 0.$$

Evaluating this expression at $\mu = R_f$,

$$\begin{aligned}
&\implies E[u''(wR_f)\alpha'(R_f)\varepsilon^2 + u'(wR_f)] = 0 \\
&\implies u''(wR_f)\alpha'(R_f)E[\varepsilon^2] + u'(wR_f) = 0 \\
&\implies u''(wR_f)\alpha'(R_f)\sigma^2 + u'(wR_f) = 0 \\
&\implies \alpha'(R_f) = -\frac{u'(wR_f)}{u''(wR_f)\sigma^2} = \frac{1}{A(wr)\sigma^2}.
\end{aligned}$$

Plugging this expression of $\alpha'(R_f)$ into (1), we get the result. ■

Question: As people get wealthier, should they invest more in the risky asset (stock)? How about the proportion of wealth invested in the risky asset?

Define $\varphi = \frac{\alpha}{w}$, the proportion of wealth invested in the risky asset. Then, we can write the consumer's problem as:

$$\max_{\varphi} E[u(wR_f + w\varphi(R - R_f))] = \max_{\varphi} E[u(w(R_f + \varphi(R - R_f))].$$

Claim 3 *Assume $u' > 0, u'' < 0, E[R] - R_f > 0$. Suppose $\boldsymbol{\gamma}(x) = -\frac{xu''(x)}{u'(x)}$(coefficient of relative risk aversion) is decreasing in $x, \forall x$. Then $\varphi'(w) \geq 0$.* OR *if $\boldsymbol{\gamma}(x)$ increasing in $x, \forall x$, then $\varphi'(w) \leq 0$.*

Exercise 2 *Does the "strict" version of the above theorem hold? That is, does $\boldsymbol{\gamma}(x)$ being strictly increasing imply $\varphi'(w) > 0$?*

Claim 4 *Same assumptions as above. Suppose $A(x)$ is decreasing in x, then $\alpha'(w) \geq 0$.*

Proof. For Claim (3): Suppose $\gamma(x)$ is decreasing in x. Because $\phi(w)$ is the optimal policy, FOC implies:

$$E[u'(w(R_f + \varphi(w)(R - R_f)))(R - R_f)] = 0 \text{ (FOC)}$$

Because the FOC must hold for all w, taking $\frac{\partial}{\partial w}$ of FOC, we have:

$$\varphi'(w) = -\frac{E[u''(c^*)(R - R_f)(R_f + \varphi(R - R_f))]}{wE[u''(c^*)(R - R_f)^2]},$$

where we denote $c^* = w(R_f + \varphi(w)(R - R_f))$ to be the consumption at the optimal policy. We want to show this expression is greater than or equal to zero. Since we know the denominator is less than zero, only need to show that

$$E[u''(c^*)(R - R_f)(R_f + \varphi(R - R_f))] > 0$$

$$\iff E[u''(c^*)(R - R_f)w(R_f + \varphi(R - R_f))] = E[u''(c^*)(R - R_f)c^*] > 0 \text{ (since } w > 0),$$

or equivalently,

$$E[\left\{\frac{u''(c^*)c^*}{u'(c^*)}\right\} u'(c^*)(R - R_f)] > 0,$$

or

$$E[-\gamma(c^*) \cdot u'(c^*)(R - R_f)] > 0.$$

On the set $R - R_f > 0$, because $\phi > 0$,

$$\begin{aligned} c^* &= w(R_f + \varphi(R - R_f)) < wR_f \\ &\Longrightarrow \boldsymbol{\gamma}(c^*) \geq \boldsymbol{\gamma}(wR_f) \\ &\Longrightarrow (R - R_f)\boldsymbol{\gamma}(c^*) \geq (R - R_f)\boldsymbol{\gamma}(wR_f), \end{aligned}$$

and on the set $R - R_f < 0$,

$$\begin{aligned} c^* &= w(R_f + \varphi(R - R_f)) > wR_f \\ &\Longrightarrow \boldsymbol{\gamma}(c^*) \leq \boldsymbol{\gamma}(wR_f) \\ &\Longrightarrow (R - R_f)\boldsymbol{\gamma}(c^*) \geq (R - R_f)\boldsymbol{\gamma}(wR_f). \end{aligned}$$

Therefore,

$$\begin{aligned}
E[u''(c^*)(R-R_f)c^*] &= E[\frac{u''(c^*)c^*}{u'(c^*)}u'(c^*)(R-R_f)] \\
&= E[-\boldsymbol{\gamma}(c^*)u'(c^*)(R-R_f)] \\
&\geq E[-\boldsymbol{\gamma}(wR_f)u'(c^*)(R-R_f)] \\
&= -\boldsymbol{\gamma}(wR_f)E[u'(c^*)(R-R_f)] = 0 \text{ (since } E[u'(c^*)(R-R_f)] = 0 \text{ by FOC).}
\end{aligned}$$

■

II Optimal Portfolio Choice with Multiple Risky Assets

A The General Case: First order expansion of FOC when risk premium is small

Suppose $u' > 0, u'' < 0$, J risky assets with random rates of return $\{R_j\}_{j=0,1,\ldots,J}$.Denote the riskfree rate $R_f = R_0$.Let $R = (R_1, R_2, ..., R_J)$ be the random vector of returns. Let $\varphi = (\varphi_1, ...\varphi_J)$,where φ_j is the proportion of wealth invested in risky asset j . Then the consumption of the agent is given by:

$$c = R_f w[1 - \sum_{j=1}^{J} \varphi_j] + \sum_{j=1}^{J} \varphi_j w R_j = w(R_f + \sum_{j=1}^{J} \varphi_j (R_j - R_f)).$$

The optimization problem of the investor can be written as

$$\max_{\varphi_1,\ldots,\varphi_J} E[u(w(R_f + \sum_{j=1}^{J} \varphi_j (R_j - R_f)))].$$

We will show that the optimal portfolio of the investor can be approximated by:

$$\varphi = \frac{R_f}{\boldsymbol{\gamma}(wR_f)} \Sigma^{-1}(\mu - R_f),$$

where $\gamma\,(w\bar{r})$ is the relative risk aversion of the utility function u at $w\bar{r}$, μ is a $J \times 1$ vector of the expected return of the risky assets. With a slight abuse of notation, I use $\mu - R_f$ to denote the $J \times 1$ vector of risk premium (In what follows, I will use R_f to denote the scalar as well as a $J \times 1$ vector, $R_f \times [1, 1, \cdots 1]^T$ interchangeably.). Finally, Σ is the $J \times J$ variance-covariance matrix of R.

Claim 5 *Suppose $\mu - R_f$ is small. Then $\varphi = \frac{R_f}{\gamma(wR_f)}\Sigma^{-1}(\mu - R_f) + o\|\mu - R_f\|$.*

Proof. We assume without proof that the optimal portfolio choice ϕ, viewed as a function of μ is C^1 (How would you prove it?). Since $\mu - R_f$ is small, we can take a first order Taylor expansion of $\varphi(\mu)$ around R_f:

$$\varphi(\mu) = \begin{pmatrix} \varphi_1(\mu_1, ..., \mu_J) \\ \varphi_2(\mu_1, ..., \mu_J) \\ \vdots \\ \varphi_J(\mu_1, ..., \mu_J) \end{pmatrix} = \varphi(R_f) + D_\mu\varphi(R_f)(\mu - R_f) + o\|\mu - R_f\| = D_\mu\varphi(R_f)(\mu - R_f) + o\|\mu - R_f\| \ (why?),$$

where I use the following notation:

$$D_\mu\varphi(\overrightarrow{r}) = \begin{pmatrix} \frac{\partial\varphi_1}{\partial\mu_1} & \frac{\partial\varphi_1}{\partial\mu_2} & \cdots & \frac{\partial\varphi_1}{\partial\mu_J} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial\varphi_J}{\partial\mu_1} & \cdots & \cdots & \frac{\partial\varphi_J}{\partial\mu_J} \end{pmatrix}.$$

Next, we use the first order conditions to show that

$$\Sigma \cdot D_\mu\varphi(R_f) = \frac{R_f}{\gamma(wR_f)} I_J,$$

where I_J is the $J-$dim identity matrix. Note we have J first order conditions with respect to φ_1, φ_2,$\cdots\varphi_J$.

$$E[u'\{w(R_f + \sum_{j=1}^{J} \varphi_j(\mu)(R_j - R_f)\}(R_i - R_f)] = 0, \forall i = 1, ..., J$$

I will write R_j as $R_j = \mu_j + \varepsilon_j$, that is, $\varepsilon_j = R_j - E[R_j]$ is the deviation of R_j from mean. The above is written as:

$$E[u'\{w(R_f + \sum_{j=1}^{J} \varphi_j(\mu)(\mu_j + \varepsilon_j - R_f)\}(\mu_i + \varepsilon_i - R_f)] = 0, \forall i = 1, ..., J.$$

Denote $c^* = w(R_f + \sum_{j=1}^{J} \varphi_j(\mu)(\mu_j + \varepsilon_j - R_f))$ to be optimal choice of consumption.

Differentiate each FOC (for $i = 1, ...J$) with respect to $\mu_k, k = 1, ...J$. For $k \neq i$, we have:

$$E[u''(c^*)w(\mu_i + \varepsilon_i - R_f)[\sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k}(\mu_j + \varepsilon_j - R_f) + \varphi_k(\mu)]] = 0.$$

This is $J(J-1)$ equations. For $k = i$:

$$E[u''(c^*)w(\mu_k + \varepsilon_k - R_f)[\sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k}(\mu_j + \varepsilon_j - R_f) + \varphi_k(\mu)] + u'(c^*)] = 0.$$

This is J equations.

Evaluate the above equations at $\mu = R_f$, we have:

$$\begin{array}{rll} \text{for } k \neq i: & E[u''(wR_f)w\varepsilon_i \sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k}\varepsilon_j] = 0 & (J(J-1) \text{ eqns}) \\ \Longrightarrow & u''(wR_f)w \sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k} E[\varepsilon_i \varepsilon_j] = 0 & \\ \Longrightarrow & \sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k}\sigma_{ij} = 0 & \end{array}$$

and

$$\begin{array}{rll} \text{for } k = i: & E[u''(wR_f)w\varepsilon_k[\sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k}\varepsilon_j] + u'(wR_f)] = 0 & (J \text{ eqns}) \\ \Longrightarrow & u''(wR_f)w \sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k} E[\varepsilon_k \varepsilon_j] + u'(wR_f) = 0 & \\ & \sum_{j=1}^J \frac{\partial \varphi_j}{\partial \mu_k}\sigma_{kj} = -\frac{u'(wR_f)}{wu''(wR_f)} & \end{array}$$

Note that the above J^2 equations can be summarized in the following matrix form:

$$\begin{pmatrix} \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1K} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \sigma_{1K} & \cdots & \cdots & \sigma_{KK} \end{pmatrix} \begin{pmatrix} \frac{\partial \varphi_1}{\partial \mu_1} & \frac{\partial \varphi_1}{\partial \mu_2} & \cdots & \frac{\partial \varphi_1}{\partial \mu_J} \\ \vdots & \ddots & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial \varphi_J}{\partial \mu_1} & \cdots & \cdots & \frac{\partial \varphi_J}{\partial \mu_J} \end{pmatrix} = \begin{pmatrix} -\frac{u'(wR_f)}{wu''(wR_f)} & 0 & \cdots & 0 \\ 0 & -\frac{u'(wR_f)}{wu''(wR_f)} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & \cdots & -\frac{u'(wR_f)}{wu''(wR_f)} \end{pmatrix},$$

that is, $\Sigma \cdot D_\mu \varphi(\overrightarrow{r}) = \frac{R_f}{\gamma(wR_f)} I_J$, which implies, $D_\mu \varphi(\overrightarrow{r}) = \frac{R_f}{\gamma(wR_f)} \Sigma^{-1}$. Substituting this into the first order Taylor expansion for $\varphi(\mu)$ gives us the result. ■

B CARA utilities and normal returns

B.1 Utility Maximization Problem

$$\max_{\alpha} \; E\left[u\left(w\bar{r}+\sum_{j=1}^{J} \alpha_j(R_j-R_f)\right)\right],$$

where $u(c)=-e^{-Ac}$ is CARA.

An important property of CARA utility is that there is no wealth effect. We have the following proposition:

Proposition 1 *Under CARA, the optimal choice of portolio holding, α^* does not depend on wealth.*

Exercise 3 *Prove this without any distributional assumptions.*

In what follows, we will focus on the case with CARA utility function ($u(c)=-e^{-Ac}$) and normally distributed returns ($R\sim N(\mu,\Sigma)$), where closed form solutions are available.

Proposition 2 *The optimal portfolio choice is*

$$\alpha^* = \frac{1}{A}\Sigma^{-1}(\mu-R_f),$$

and the expected utility of the agent conditioning on the belief $N(\mu,\Sigma)$ is

$$V(w|\,\mu,\Sigma) = -e^{-Aw\bar{r}-\frac{1}{2}(\mu-R_f)\Sigma^{-1}(\mu-R_f)}.$$

Proof. To be added. ■

B.2 The value of signals

Suppose the investor can observe a signal on R of the form $s=R+\varepsilon$, where $\varepsilon\perp R$ and $\varepsilon\sim N(0,\Delta)$. The Bayes rule implies that

$$E[R|\,s] = m = \Omega\left[\Sigma^{-1}\mu+\Delta^{-1}s\right],$$

where

$$Var[R|\,s] = \Omega = \left[\Sigma^{-1}+\Delta^{-1}\right]^{-1}$$

Also, before observing the signal, the unconditional expectation of m is $E[m]=\mu$, $Var[m]=\Sigma-\Omega$ (Prove this). In the homework, you are asked to answer the following questions:

Howework Exercise:

1. Suppose the agent is allowed to rebalance his portfolio after observing the signal, how much is he willing to pay for the signal?

2. Suppose he is not allowed to rebalance his portfolio after observing the signal, how much is he willing to pay for the signal?

3. Consider two investor with different absolute risk aversion $A_1 < A_2$. Who is willing to pay for a higher price for the signal? why?

Assume you can rebalance your portfolio after observing the signal. How much will you pay for the signal? (the next proposition will be useful)

Answer: You will pay up to $\frac{1}{2\sigma\bar{r}}\left[\log|\Sigma| - \log|\Omega|\right]$, and after the signal then $\alpha(s) = \frac{1}{A}\Omega^{-1}\left[m\left(s\right) - r\right]$.

Proposition 3 *Let the prior of* R *be* $N(\vec{\mu}, \Sigma)$*. If we observe* $s = R + \varepsilon$ *(where* $\varepsilon \perp R$ *and* $\varepsilon \sim N(0, \Delta)$ *) then the posterior distribution is* $N(m, \Omega)$*, where*

$$\Omega = \left(\Sigma^{-1} + \Delta^{-1}\right)^{-1} \qquad \text{(Inverse of the information matrix)}$$
$$m = \Omega\left(\Sigma^{-1}\mu + \Delta^{-1}s\right)$$

Exercise 4 *Now suppose the investor is not allowed to rebalance. How much will you pay?*