ONLINE APPENDIX - Proofs

Proof of Proposition 1: Using the results in Duffie and Epstein (1992b), the Hamilton-Jacobi-Bellman (HJB) equation for the planner problem in (31) is written as

$$\max_{C} \{ f(C, V(m,K)) + \mathcal{L}V(m,K) \} = 0, \tag{IA.1}$$

where $\mathcal{L}$ is the differential operator associated with the process $\{K_t\}$ and $\{m_t\}$, i.e.,

$$\begin{aligned} \mathcal{L}V(m,K) &= (Km - c) V_K + \frac{1}{2} K^2 \sigma_K^2 V_{KK} \\ &\quad + a\left(\overline{\theta} - m\right) V_m + \frac{1}{2}\sigma_m^2 V_{mm} + (\rho \sigma_K \sigma_\theta + Q) K V_{mK}. \end{aligned}$$

By homogeneity, V must be of the form in (32). Monotonicity of the value function with respect to m implies that $H(m; \sigma_e)$ is strictly increasing (decreasing) in m if $\gamma < 1 (\gamma > 1)$. Using (32) to simplify the HJB equation, and denoting $x = \frac{K}{C}$, I have that

$$\begin{aligned} \max_x \left\{ \frac{\beta}{1 - 1/\psi} x^{\frac{1}{\psi} - 1} H(m; \sigma_e)^{1 - \frac{1-1/\psi}{1-\gamma}} + \left(m - \frac{1}{x} - \frac{1}{2}\gamma\sigma_K^2 \right) H(m; \sigma_e) \right\} \\ - \frac{\beta}{1 - 1/\psi} H(m; \sigma_e) + \left[\frac{1}{1-\gamma} a\left(\overline{\theta} - m\right) + (\rho\sigma_K\sigma_\theta + Q) \right] H'(m; \sigma_e) \\ + \frac{1}{2}\frac{1}{1-\gamma}\sigma_m^2 H''(m; \sigma_e) = 0. \end{aligned} \tag{IA.2}$$

The first-order condition of the HJB equation in (IA.2) implies the optimal consumption policy function is of the form in (34), and $x(m; \sigma_e)$ is the wealth-consumption ratio given by (35). Using the optimal consumption policy, the HJB equation in (IA.2) can be simplified to an ODE of $H(m;\sigma_e)$:

$$\frac{1}{\psi-1}\beta^{\psi}H(m;\sigma_e)^{\frac{1-\psi}{1-\gamma}}+\left(m-\frac{\beta}{1-1/\psi}-\frac{1}{2}\gamma\sigma_K^2\right) \tag{IA.3}$$
$$+\left[\frac{1}{1-\gamma}a\left(\overline{\theta}-m\right)+\left(\rho\sigma_K\sigma_\theta+Q\right)\right]\frac{H'(m;\sigma_e)}{H(m;\sigma_e)}+\frac{1}{2}\frac{1}{1-\gamma}\sigma_m^2\frac{H''(m;\sigma_e)}{H(m;\sigma_e)}=0.$$

Finally, the monotonicity of $H(m;\sigma_e)$ implies that $H(m;\sigma_e)^{\frac{1}{1-\gamma}}$ is always a strictly increasing function of m. Consequently, the wealth-consumption ratio function in (35) is strictly increasing (decreasing) in m if $\psi>1$ ($\psi<1$).

The State Price Density in Lemma 1: As shown by Duffie and Epstein (1992a), in an economy with recursive preferences, the state price density, denoted $\{\pi_t\}_{t\geq 0}$, satisfies

$$\frac{d\pi_t}{\pi_t}=\frac{df_C(C_t,V_t)}{f_C(C_t,V_t)}+f_V(C_t,V_t)\,dt. \tag{IA.4}$$

Using equation (39) in Lemma 1,

$$\frac{d\pi_t}{\pi_t}-E_t\left[\frac{d\pi_t}{\pi_t}\right]=\frac{dU_W(W_t,m_t)}{U_W(W_t,m_t)}-E_t\left[\frac{dU_W(W_t,m_t)}{U_W(W_t,m_t)}\right]. \tag{IA.5}$$

Equation (43) can then be obtained by applying Ito's formula to (IA.5).

The Log-Linearization Approximation of the Value Function: I now derive a closed form expression for the log-linearization approximation of equation (48) using the method proposed by Campbell, Chacko, Rodriguez, and Viceira (2004). Note the only nonlinear term in (IA.3) is $\beta^{\psi}H(m)^{\frac{1-\psi}{1-\gamma}}$, which is exactly the inverse of the wealth-consumption ratio function $x(m;\sigma_e)$. Using a log-linear approximation of $x(m;\sigma_e)^{-1}$, I have

$$x(m)^{-1} \approx \kappa_0 + \kappa_1 \log x(m;\sigma_e) = \kappa_0 + \kappa_1 \psi \ln\beta + \kappa_1 \frac{1-\psi}{1-\gamma} \ln H(m;\sigma_e), \tag{IA.6}$$

where

$$\kappa_0 = \kappa_1 (1 - \log \kappa_1) \tag{IA.7}$$

and κ_1 can be chosen as the consumption-wealth ratio when m is equal to its unconditional mean $\bar{\theta}$:

$$\kappa_1 = x\left(\bar{\theta};\sigma_e\right).$$

Using (IA.6), one can approximate the ODE in (IA.3) as

$$\begin{aligned} &\frac{1}{\psi-1}\left[\kappa_0 + \kappa_1\psi\log\beta + \kappa_1\frac{1-\psi}{1-\gamma}\log H(m;\sigma_e)\right] + \left(m - \frac{\beta}{1-1/\psi} - \frac{1}{2}\gamma\sigma_K^2\right) \\ &\quad + \left[\frac{1}{1-\gamma}a\left(\bar{\theta}-m\right) + (\rho\sigma_K\sigma_\theta + Q)\right]\frac{H'(m;\sigma_e)}{H(m;\sigma_e)} + \frac{1}{2}\frac{1}{1-\gamma}\sigma_m^2\frac{H''(m;\sigma_e)}{H(m;\sigma_e)} = 0. \end{aligned} \tag{IA.8}$$

I guess $H(m;\sigma_e)$ is of the form

$$H(m;\sigma_e) = \exp(A + Bm). \tag{IA.9}$$

Using the method of undetermined coefficients, I have that

$$B = \frac{1-\gamma}{a+\kappa_1} \tag{IA.10}$$

and

$$A = \frac{1}{\kappa_1}\left\{\frac{1}{2}\sigma_m^2 B^2 + \left[(1-\gamma)(\rho\sigma_K\sigma_\theta + Q) + a\bar{\theta}\right]B + D\right\}, \tag{IA.11}$$

where

$$D = \frac{1-\gamma}{1-\psi}\left[-\kappa_0 - \psi\kappa_1 \log\beta + \psi\beta - \frac{1}{2}\gamma\left(1-\psi\right)\sigma_K^2\right].$$

Using (IA.9),

$$\frac{H'\left(m;\sigma_e\right)}{H\left(m;\sigma_e\right)} = B = \frac{1-\gamma}{a+\kappa_1}. \tag{IA.12}$$

Therefore, the risk premium can be approximated by

$$\mu_{Rt} - r_t \approx \gamma\sigma_K^2 + \frac{\gamma-1}{a+\kappa_1}\left(\rho\sigma_K\sigma_\theta + Q\right). \tag{IA.13}$$

This above approximation is exact if $\psi = 1$, in which case

$$B = \frac{1-\gamma}{a+\beta} \tag{IA.14}$$

and

$$A = \frac{1}{\kappa_1}\left\{\frac{1}{2}\sigma_m^2 B^2 + \left[\left(1-\gamma\right)\left(\rho\sigma_K\sigma_\theta + Q\right) + a\overline{\theta}\right]B + \left(1-\gamma\right)\left(\beta\log\beta - \beta - \frac{1}{2}\gamma\sigma_K^2\right)\right\}. \tag{IA.15}$$

The Equity Premium in the Pure Exchange Economy: First, I consider the solution to the Kalman filter of the agent's learning problem. Learning steady-state implies that $m_t = E_t\left[\theta_t\right]$, and m_t satisfies equation (61), where

$$Q_E = \frac{\left(1-\rho^2\right)\sigma_\theta^2}{\left(a+\rho\frac{\sigma_\theta}{\sigma_Y}\right) + \sqrt{\left(a+\rho\frac{\sigma_\theta}{\sigma_Y}\right)^2 + \left(1-\rho\right)\sigma_\theta^2\left(\sigma_Y^{-2}+\sigma_e^{-2}\right)}}. \tag{IA.16}$$

Also, theorem 8.1 in Lipster and Shiryaev (2001) implies

$$Cov_t\left[dm_t, \frac{dY_t}{Y_t}\right] = (\rho\sigma_Y\sigma_\theta + Q_E),$$

and

$$Var_t[dm_t] = \sigma_\theta^2 - 2aQ_E.$$

Second, I derive the ODE that the function $G(m)$ in equation (62) has to satisfy. To solve the equilibrium optimal portfolio choice problem of the representative agent, it is enough to consider the case in which there are two assets, a risk-free bond, and an equity that pays aggregate consumption as its dividends. From Duffie and Epstein (1992a), the value function of the optimal portfolio choice problem satisfies

$$\max_{C,\phi}\left\{f(C, U(W,m)) + \mathcal{L}^{C,\phi}U(W,m)\right\} = 0, \tag{IA.17}$$

where $\mathcal{L}^{C,\phi}$ is the differential operator under the policy $\{C, \phi\}$. Using the law of motion of W and m in equation (38) and (61), the above is written as

$$\max_{C,\phi}\left\{\begin{array}{c} f(C, U(W,m)) + U_W\left[W\left[\phi\mu_{R,W}(m) + (1-\phi)r(m)\right] - C\right] \\ +W\phi\sigma_{W,m}(m)U_{W,m} + \frac{1}{2}W^2\phi^2\sigma_{R,W}^2(m)U_{WW} \end{array}\right\}$$
$$+a\left(m - \bar{\theta}\right)U_m + \frac{1}{2}\sigma_m^2(m)U_{mm} = 0, \tag{IA.18}$$

where I use the notation

$$\sigma_{W,m}(m_t) = Cov_t(dR_{W,t}, dm_t),$$

$$\sigma_{R,W}^2(m_t) = Var_t[dR_{W,t}],$$

and

$$\sigma_m^2(m_t) = Var_t[dm_t].$$

Here I assume the drift and diffusion coefficients of the returns are all functions of the state variable m_t, which can be verified after all equilibrium prices and quantities are constructed.

By equation (63), and condition (39) in Lemma 1, the equilibrium consumption and wealth satisfies

$$\beta^{\psi} G(m)^{-\frac{1-1/\psi}{1-\gamma}} W = C = Y, \tag{IA.19}$$

where the second equality is due to the market clearing condition. Applying Ito's Lemma, and using the definition of cumulative return in equation (15), I have

$$\begin{aligned} \mu_{R,W}(m) &= m + \frac{1-\frac{1}{\psi}}{1-\gamma}\left[a\left(\bar{\theta}-m\right)+\rho\sigma_Y\sigma_\theta+Q_E\right]\frac{G'(m)}{G(m)} \\ &+\frac{1-\frac{1}{\psi}}{1-\gamma}\left[\frac{\gamma-\frac{1}{\psi}}{1-\gamma}\left(\frac{G'(m)}{G(m)}\right)^2+\frac{G''(m)}{G(m)}\right]\left[\sigma_\theta^2-2aQ_E\right] \\ &+\beta^{\psi}G(m)^{-\frac{1-1/\psi}{1-\gamma}}, \end{aligned} \tag{IA.20}$$

$$\sigma_{R,W}^2(m) = \left(\frac{1-\frac{1}{\psi}}{1-\gamma}\right)^2\left(\frac{G'(m)}{G(m)}\right)^2\left[\sigma_\theta^2-2aQ_E\right]+\sigma_Y^2+2\frac{1-\frac{1}{\psi}}{1-\gamma}\frac{G'(m)}{G(m)}\left[\rho\sigma_Y\sigma_\theta+Q_E\right], \tag{IA.21}$$

and

$$\sigma_{W,m}(m) = \frac{1-\frac{1}{\psi}}{1-\gamma}\frac{G'(m)}{G(m)}\left[\sigma_\theta^2-2aQ_E\right]+\rho\sigma_Y\sigma_\theta+Q_E \tag{IA.22}$$

Using equation (IA.19) - (IA.22) and noting that $\phi = 1$ in equilibrium, equation (IA.18) can be simplified to the following ODE:

$$\frac{\beta^{\psi}}{1-1/\psi}G(m)^{-\frac{1-1/\psi}{1-\gamma}}+\left(m-\frac{\beta}{1-1/\psi}-\frac{1}{2}\gamma\sigma_Y^2\right) \tag{IA.23}$$
$$+\left[\frac{1}{1-\gamma}a\left(\overline{\theta}-m\right)+\left(\rho\sigma_Y\sigma_\theta+Q_E\right)\right]\frac{G'(m)}{G(m)}+\frac{1}{2}\frac{1}{1-\gamma}\left[\sigma_\theta^2-2aQ_E\right]\frac{G''(m)}{G(m)}=0.$$

Using log-linear approximation, it can be shown

$$\frac{G'(m)}{G(m)}\approx\frac{1-\gamma}{a+\varpi_1}, \tag{IA.24}$$

where ϖ_1 is the steady-state consumption-wealth ratio in the exchange economy.

Third, I derive the expressions for the myopic demand component and the hedging demand component of the risk premium on aggregate wealth. I use the definition of myopic demand component and hedging demand component of equity premium in equations (63) and (64) to obtain

$$MD_t=\gamma\left[\sigma_Y^2+2\frac{1-\frac{1}{\psi}}{1-\gamma}\frac{G'(m_t)}{G(m_t)}\left(\rho\sigma_Y\sigma_\theta+Q_E\right)+\left(\frac{1-\frac{1}{\psi}}{1-\gamma}\right)^2\left(\frac{G'(m_t)}{G(m_t)}\right)^2\left(\sigma_\theta^2-2aQ_E\right)\right] \tag{IA.25}$$

and

$$HD_t=-\frac{1}{\psi}\frac{G'(m_t)}{G(m_t)}\left[\frac{1-\frac{1}{\psi}}{1-\gamma}\frac{G'(m_t)}{G(m_t)}\left(\sigma_\theta^2-2aQ_E\right)+\left(\rho\sigma_Y\sigma_\theta+Q_E\right)\right]. \tag{IA.26}$$

Finally, I show that in the special case of CRRA, the effect on the myopic demand component of the equity premium always dominates. Note that equation (IA.25) and (IA.26) imply that

$$\frac{\partial}{\partial\sigma_e}MD_t\approx 2\gamma\left(1-\frac{1}{\psi}\right)\frac{\left(\varpi_1+a/\psi\right)}{\left(a+\varpi_1\right)^2}\frac{\partial Q}{\partial\sigma_e} \tag{IA.27}$$

and

$$\frac{\partial}{\partial \sigma_e} HD_t \approx \frac{1}{\psi}(\gamma - 1)\frac{[a(2/\psi - 1) + \kappa_1]}{(a + \varpi_1)^2}\frac{\partial Q}{\partial \sigma_e}. \tag{IA.28}$$

Note that for ψ close to 1, the sign of (IA.27) is determined by $1 - \frac{1}{\psi}$, and the sign of (IA.28) is determined by $\gamma - 1$. If I impose $\gamma = \frac{1}{\psi}$, then

$$\frac{\partial}{\partial \sigma_e}[\mu_{Rt} - r_t] = \frac{\partial}{\partial \sigma_e} MD_t + \frac{\partial}{\partial \sigma_e} HD_t \approx \left(1 - \frac{1}{\psi}\right)\frac{\frac{1}{\psi}}{a + \varpi_1}\frac{\partial Q}{\partial \sigma_e} \tag{IA.29}$$

Note that (IA.29) will have the same sign as (IA.27), but opposite to (IA.28). Therefore, the effect through the myopic demand component dominates.