Solving SDU with the Markov Chain Approximation Method [1]

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Abstract

[1] This note details the Markov Chain Approximation method I use to solve the social planner's problem in the production economy with stochastic differential utility in Ai (2010).

Introduction

This is the notes of the Markov Chain Approximation method that I use to solve the model in Ai (2010). I adapt the Markov Chain Approximation method of Kushner and Dupuis to solve the optimal control problem in Ai (2010). The easies way to understand this approach is to consider a discretization of the continuous time problem.

I Discretization of Recursive Preference

Denote $s_t = (K_t, m_t)$ the vector of state variables. We consider the following discretization of the dynamic programing problem considered in the paper:

$$V(s_t) = \left\{ \left(1 - e^{-\beta\Delta}\right) C(s_t)^{1-\frac{1}{\psi}} + e^{-\beta\Delta} \left(E\left[V(s_{t+1}) \middle| s_t\right]\right)^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1-\gamma}{1-1/\psi}}, \tag{1}$$

where $C(s_t)$ denote the optimal consumption given the current state s_t. The above equation can be written as:

$$V(s_t)^{\frac{1-1/\psi}{1-\gamma}} = \left(1 - e^{-\beta\Delta}\right) C(s_t)^{1-\frac{1}{\psi}} + e^{-\beta\Delta} \left(E\left[V(s_{t+\Delta}) \middle| s_t\right]\right)^{\frac{1-1/\psi}{1-\gamma}} \tag{2}$$

When Δ is small, $E\left[V(s_{t+\Delta}) \middle| s_t\right]$ can be approximated by

$$V(s_t) + \mathcal{L}V(s_t)\Delta,$$

where

$$\mathcal{L}V(s_t) = \lim_{\Delta \to 0} \frac{1}{\Delta} E\left[V(s_{t+\Delta}) - V(s_t) \middle| s_t\right]$$

Therefore, for small Δ, the term $\left(E\left[V(s_{t+\Delta}) \middle| s_t\right]\right)^{\frac{1-1/\psi}{1-\gamma}}$ can be approximated by:

$$\begin{aligned} \left(E\left[V(s_{t+\Delta}) \middle| s_t\right]\right)^{\frac{1-1/\psi}{1-\gamma}} &= \left[V(s_t) + \mathcal{L}V(s_t)\Delta\right]^{\frac{1-1/\psi}{1-\gamma}} + o\|\Delta\| \\ &\approx V(s_t)^{\frac{1-1/\psi}{1-\gamma}} + \frac{1-1/\psi}{1-\gamma} V(s_t)^{\frac{1-1/\psi}{1-\gamma}-1} \mathcal{L}V(s_t)\Delta + o\|\Delta\| \end{aligned} \tag{3}$$

Hence the recursion (2) can be writtens as:

$$V(s_t)^{\frac{1-1/\psi}{1-\gamma}} = \left(1 - e^{-\beta\Delta}\right) C(s_t)^{1-\frac{1}{\psi}} + e^{-\beta\Delta} \left[V(s_t)^{\frac{1-1/\psi}{1-\gamma}} + \frac{1-1/\psi}{1-\gamma} V(s_t)^{\frac{1-1/\psi}{1-\gamma}-1} \mathcal{L}V(s_t)\Delta\right] + o\|\Delta\|. \tag{4}$$

For Δ small, Equation (4) implies the following iteration procedure:

$$V_{n+1}(s_t)^{\frac{1-1/\psi}{1-\gamma}} = \left(1-e^{-\beta\Delta}\right) C_n(s_t)^{1-\frac{1}{\psi}} + e^{-\beta\Delta}\left[V_n(s_t)^{\frac{1-1/\psi}{1-\gamma}} + \frac{1-1/\psi}{1-\gamma} V_n(s_t)^{\frac{1-1/\psi}{1-\gamma}-1}\mathcal{L}V_n(s_t)\Delta\right],$$

where C_n denotes the optimal consumption choice given the value function is V_n. Divide both sides of the equation by $V_n(t)^{\frac{1-1/\psi}{1-\gamma}}$, we have:

$$\left(\frac{V_{n+1}(s_t)}{V_n(s_t)}\right)^{\frac{1-1/\psi}{1-\gamma}} = \left(1-e^{-\beta\Delta}\right) C_n(s_t)^{1-\frac{1}{\psi}} V_n(s_t)^{-\frac{1-1/\psi}{1-\gamma}} + e^{-\beta\Delta}\left[1 + \frac{1-1/\psi}{1-\gamma} V_n(s_t)^{-1}\mathcal{L}V_n(s_t)\Delta\right] \tag{5}$$

Note, the value function is homogenous of degree $1-\gamma$ in K. Consistent with the notation in the paper, we denote

$$V_n(s_t) = H_n(m) K^{1-\gamma}.$$

Using the optimal policy function,

$$C_n(s_t) = \beta^{\psi} H_n(m_t)^{\frac{1-\psi}{1-\gamma}} K_t;$$

therefore,

$$C(s_t)^{1-\frac{1}{\psi}} V_n(s_t)^{-\frac{1-1/\psi}{1-\gamma}} = \beta^{\psi-1} H_n(m_t)^{\frac{1-\psi}{1-\gamma}}.$$

Now, equation (5) can be written as:

$$\left(\frac{V_{n+1}(s_t)}{V_n(s_t)}\right)^{\frac{1-1/\psi}{1-\gamma}} = \beta^{\psi} H_n(m_t)^{\frac{1-\psi}{1-\gamma}}\Delta + e^{-\beta\Delta}\left[1 + \frac{1-1/\psi}{1-\gamma} V_n(s_t)^{-1}\mathcal{L}V_n(s_t)\Delta\right] \tag{6}$$

II Markov Chain Approximation

Lemma 1 *Consider the following two-dimensional diffusion process:*

$$d\begin{bmatrix} x_{1,t} \\ x_{2,t} \end{bmatrix} = \begin{bmatrix} b_1(x_t) \\ b_2(x_t) \end{bmatrix} dt + \begin{bmatrix} \sigma_{1,1}(x_t) & \sigma_{1,2}(x_t) \\ \sigma_{2,1} & \sigma_{2,2}(x_t) \end{bmatrix} d\begin{bmatrix} W_{1,t} \\ W_{2,t} \end{bmatrix}.$$

The following is a locally consistent MC approximation of the diffusion process. At time t, the transition probability from t to $t+\Delta$, with $\Delta = h^2/Q^h(x_t)$ is:

$$\begin{aligned}
P\left(x_t, x_t \pm h v_1(x_t)\right) &= \frac{1}{2} q_1^2 Q^h(x_t)^{-1} \\
P\left(x_t, x_t \pm h v_2(x_t)\right) &= \frac{1}{2} q_2^2 Q^h(x_t)^{-1} \\
P\left(x_t, x_t \pm \frac{1}{\alpha_i} h e_i\right) &= \alpha_i b_i(x_t)^{\pm} h Q^h(x_t)^{-1}, \quad i = 1, 2,
\end{aligned}$$

where we use the following notation:

$$Q^h(x) = q_1^2 + q_2^2 + h\left[|b_1(x)| + |b_2(x)|\right],$$

$$e_1 = [1, 0]^T; \quad e_2 = [0, 1]^T,$$

$$v_1(x) = q_1^{-1}\left[\sigma_{1,1}(x), \sigma_{2,1}(x)\right]^T; \qquad v_2(x) = q_2^{-1}\left[\sigma_{1,2}(x), \sigma_{2,2}(x)\right]^T.$$

Proof. *To be added.* ■

Considder the law of motion of state varibles in our model:

$$\begin{aligned}
dK_t &= K_t\left\{[m_t - c_t]\, dt + \sigma_K d\tilde{B}_{K,t}\right\} \\
dm_t &= a\left(\bar{\theta} - m_t\right) dt + R\left[\sigma_K^{-1} d\tilde{B}_{K,t} + \sigma_e^{-1} d\tilde{B}_{e,t}\right]
\end{aligned}$$

They can be written as:

$$d\begin{bmatrix} K \\ m \end{bmatrix} = \begin{bmatrix} K(m-c) \\ a\left(\bar{\theta} - m\right) \end{bmatrix} dt + \begin{bmatrix} K\sigma_K \\ R\sigma_K^{-1} \end{bmatrix} \tilde{B}_{K,t} + \begin{bmatrix} 0 \\ R\sigma_e^{-1} \end{bmatrix} \tilde{B}_{e,t}.$$

Using the above lemma, we can construct the following locally consistent approximation of the Markov Chain:

$$P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K \\ m \end{bmatrix} \pm hq_1^{-1} \begin{bmatrix} K\sigma_K \\ R\sigma_K^{-1} \end{bmatrix}\right) = \frac{1}{2}q_1^2 Q^{-1} \tag{7}$$

$$P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K \\ m \end{bmatrix} \pm hq_2^{-1} \begin{bmatrix} 0 \\ R\sigma_e^{-1} \end{bmatrix}\right) = \frac{1}{2}q_2^2 Q^{-1} \tag{8}$$

$$P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K \\ m \end{bmatrix} \pm h\frac{1}{\alpha_1} \begin{bmatrix} K\sigma_K \\ R\sigma_K^{-1} \end{bmatrix}\right) = \alpha_1 (m-c)^{\pm} hQ^{-1} \tag{9}$$

$$P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K \\ m \end{bmatrix} \pm h\frac{1}{\alpha_2} \begin{bmatrix} K\sigma_K \\ R\sigma_K^{-1} \end{bmatrix}\right) = \alpha_2 a \left(\bar{\theta}-m\right)^{\pm} hQ^{-1} \tag{10}$$

Note the Markov Chain approximation is locally consistent for any choice of α and q. We choose $\alpha_2 = 1$; $q_1 = R\sigma_K^{-1}$; $q_2 = R\sigma_e^{-1}$ and $\alpha_1 = q_1\sigma_K^{-1}$. This choice of the α and q minimizes evaluation of the function H and saves computational power. In this case, the above Markov Chain becomes:

$$\begin{aligned}
P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K\left(1 \pm q_1^{-1}\sigma_K h\right) \\ m \pm h \end{bmatrix}\right) &= \frac{1}{2}q_1^2 Q^{-1} \\
P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K \\ m \pm h \end{bmatrix}\right) &= \frac{1}{2}q_2^2 Q^{-1} \\
P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K\left(1 \pm q_1^{-1}\sigma_K h\right) \\ m \end{bmatrix}\right) &= q_1\sigma_K^{-1} (m-c)^{\pm} hQ^{-1} \\
P\left(\begin{bmatrix} K \\ m \end{bmatrix}, \begin{bmatrix} K \\ m \pm h \end{bmatrix}\right) &= a\left(\bar{\theta}-m\right)^{\pm} hQ^{-1},
\end{aligned}$$

where q_1 and q_2 are given as above, and $Q = q_1^2 + q_2^2 + h\left[q_1\sigma_K^{-1}|m-c| + a\left|\bar{\theta}-m\right|\right]$.

III The Term $\mathcal{L}V_n(s_t)\Delta$

The term $\mathcal{L}V_n(s_t)\Delta$ can be approximated by:

$$\mathcal{L}V_n(s_t)\Delta = E\left[H(m_{t+\Delta})K_{t+\Delta}^{1-\gamma}\right] - H(m_t)K_t^{1-\gamma}$$

Using the Markov Chain approximation of $[K_t, m_t]$, we have:

$$\begin{aligned}
E\left[H\left(m_{t+\Delta}\right) K_{t+\Delta}^{1-\gamma}\right] &= \frac{1}{2} q_1^2 Q^{-1} \times K^{1-\gamma}\left(1 \pm q_1^{-1} \sigma_K h\right)^{1-\gamma} H\left(m \pm h\right) \\
&\quad + \left[\frac{1}{2} q_2^2 Q^{-1} + a\left(\bar{\theta} - m\right)^{\pm} h Q^{-1}\right] K^{1-\gamma} H\left(m \pm h\right) \\
&\quad + q_1 \sigma_K^{-1}\left(m - c\right)^{\pm} h Q^{-1} \times K^{1-\gamma}\left(1 \pm q_1^{-1} \sigma_K h\right)^{1-\gamma} H\left(m\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathcal{L} V_n\left(s_t\right) \Delta &= E\left[H\left(m_{t+\Delta}\right) K_{t+\Delta}^{1-\gamma}\right] - H\left(m_t\right) K_t^{1-\gamma} \\
&= K^{1-\gamma}\left\{\begin{array}{c}
\frac{1}{2} q_1^2 Q^{-1}\left(1 \pm q_1^{-1} \sigma_K h\right)^{1-\gamma} H\left(m \pm h\right) \\
+\left[\frac{1}{2} q_2^2 Q^{-1} + a\left(\bar{\theta} - m\right)^{\pm} h Q^{-1}\right] H\left(m \pm h\right) \\
+q_1 \sigma_K^{-1}\left(m - c\right)^{\pm} h Q^{-1}\left(1 \pm q_1^{-1} \sigma_K h\right)^{1-\gamma} H\left(m\right) - H\left(m\right)
\end{array}\right\}.
\end{aligned}$$

The term $V_n\left(s_t\right)^{-1} \mathcal{L} V_n\left(s_t\right) \Delta$ in Equation (6) can then be written as:

$$\begin{array}{c}
Term = \frac{1}{2} q_1^2 Q^{-1}\left(1 \pm q_1^{-1} \sigma_K h\right)^{1-\gamma} \frac{H(m \pm h)}{H(m)} \\
+\left[\frac{1}{2} q_2^2 Q^{-1} + a\left(\bar{\theta} - m\right)^{\pm} h Q^{-1}\right] \frac{H(m \pm h)}{H(m)} \\
+q_1 \sigma_K^{-1}\left(m - c\right)^{\pm} h Q^{-1}\left(1 \pm q_1^{-1} \sigma_K h\right)^{1-\gamma} - 1
\end{array} \tag{11}$$

IV The "Bellman" Operator

We have the following operator that maps $H_n\left(m\right)$ to $H_{n+1}\left(m\right)$:

$$\left(\frac{H_{n+1}\left(m_t\right)}{H_n\left(m_t\right)}\right)^{\frac{1-1/\psi}{1-\gamma}} = \beta^{\psi} H_n\left(m_t\right)^{\frac{1-\psi}{1-\gamma}} \Delta + e^{-\beta \Delta}\left[1 + \frac{1 - 1/\psi}{1-\gamma} Term\right].$$

Take log on both sides, and denote $h\left(m\right) = \ln H\left(m\right)$, we have:

$$h_{n+1}\left(m\right) = h_n\left(m\right) + \frac{1-\gamma}{1 - 1/\psi} \ln\left\{\beta^{\psi} e^{\frac{1-\psi}{1-\gamma} h_n(m)} \Delta + e^{-\beta \Delta}\left[1 + \frac{1 - 1/\psi}{1-\gamma} Term\right]\right\},$$

where

$$
\begin{aligned}
Term &= \frac{1}{2}q_1^2 Q^{-1}\left(1 \pm q_1^{-1}\sigma_K h\right)^{1-\gamma}\exp\left\{h_n\left(m \pm h\right) - h_n\left(m\right)\right\} \\
&+ \left[\frac{1}{2}q_2^2 Q^{-1} + a\left(\bar{\theta} - m\right)^{\pm} hQ^{-1}\right]\exp\left\{h_n\left(m \pm h\right) - h_n\left(m\right)\right\} \\
&+ q_1\sigma_K^{-1}\left(m - c\right)^{\pm} hQ^{-1}\left(1 \pm q_1^{-1}\sigma_K h\right)^{1-\gamma} - 1
\end{aligned}
$$

Numerically, I choose a large enough interval so that the probability of m appear outside of the interval is small under the steady-state distribution. I iterate the above operator until convergence.

V Reference

Ai, Hengjie (2010), Information Quality and Long-run Risk: Asset Pricing Implications, The Journal of Finance, Vol LXV, No. 4, August, 2010.

Kushner, Harold J., and Paul G. Dupuis, 2001, NumericalMethods for Stochastic Control Problems in Continuous Time (Springer-Verlag, New York).