Toward a Quantitative General Equilibrium Asset Pricing Model with Intangible Capital

Hengjie Ai, Mariano Massimiliano Croce, and Kai Li[1]

Abstract

We model investment options as intangible capital in a production economy in which younger vintages of assets in place have lower exposure to aggregate productivity risk. In equilibrium, physical capital requires a substantially higher expected return than intangible capital. Quantitatively, our model rationalizes a significant share of the observed difference in the average return of book-to-market-sorted portfolios (value premium). Our economy also produces (1) a high premium of the aggregate stock market over the risk-free interest rate, (2) a low and smooth risk-free interest rate, and (3) key features of the consumption and investment dynamics in the US data.

Introduction

Historically, stocks with high book-to-market ratios, i.e., value stocks, earn a higher average return than those with low book-to-market ratios, i.e., growth stocks (Fama et al. (1992, 1995)). The difference in log units is approximately 4.3% per year and is known as the value premium. The market-to-book ratio of a firm is often viewed as a measure of the intensity of future growth options relative to assets currently in place. Interpreted this way, the empirical evidence on value premium suggests that the average spread between the return on physical assets in place and growth options is comparable to the aggregate stock market equity premium.

In this paper we propose a quantitative general equilibrium model in which growth options form intangible capital. When calibrated to standard statistics of the dynamics of macroeconomic quantities, our model is able to reproduce key features of asset returns data, including the difference in the average return on installed physical capital and future growth opportunities. Our model generates a high equity premium (5.66% per year for the market return, in log units) with a moderate risk aversion of 10 and a low and smooth risk-free interest rate. Our results are comparable to those obtained by the standard real business cycle (RBC) models in terms of the second moments of aggregate consumption, investment, and hours worked. Furthermore, the expected annual log return on growth options is 4.08% lower than that on installed physical capital, a significant share of the observed value premium in the data.

We follow Ai (2009) and model growth options as intangible capital in an otherwise standard neoclassical production economy. In contrast to assets in place, growth options do not produce consumption goods, and hence their payoff is not directly linked to aggregate productivity shocks. Rather, they represent an investment opportunity that allows their owner to build new production units using physical investment goods. Higher aggregate investment enables a greater fraction of growth options to be implemented and yield a higher payoff. Thus, in our model, the returns of growth options and physical capital depend on different risk factors, and hence feature different risk premiums in equilibrium.

We make two major modifications to the Ai (2009) model. First, we adopt recursive preferences and an aggregate productivity process with long-run risk as in Croce (2008). This allows us to generate a highly volatile pricing kernel. More importantly, we show that in our model physical capital endogenously has a much higher exposure to long-run risk than intangible capital. Our production-based model thus rationalizes the empirical findings on the cross-section of equity returns in Bansal et al. (2005), Hansen et al. (2008), and Kiku (2006).

Second, focusing on US microeconomic data we document that the productivity of new vintages of capital is less sensitive to aggregate productivity shocks than that of older vintages. Based on this novel empirical finding, our model features heterogeneous productivity of vintage capital, with young vintages having lower exposure to aggregate shocks, as in the data. As a result, in our economy the response of physical investment with respect to unexpected fluctuations in aggregate productivity (short-run shocks) is positive, as in standard RBC models, but it is negative with respect to news about future productivity shocks (long-run shocks). These findings provide a crucial explanation of the high equity premium, large spread between the return on growth options and assets in place, and significant volatility of investment observed in the data.

In our setup, the elasticity of substitution between tangible investment and intangible capital is high, implying that the adjustment of tangible capital is not costly. Consequently, investment responds strongly to contemporaneous productivity shocks, as it does in standard RBC models. The response of investment to long-run shocks, however, is sluggish for two reasons. First, news shocks predict future productivity growth but do not affect current output. Because of consumption-smoothing motives, the agent tends to avoid dramatic changes in investment, as they cause fluctuations in consumption in the opposite direction. Second, since new investments are less exposed to aggregate shocks due to their young age, their productivity is affected by news shocks only with a delay. The agent, therefore, finds it optimal to postpone the adjustment of investment with respect to such shocks. In equilibrium, after a long-run productivity shock, the price of physical capital responds immediately and sharply, whereas physical investment and the return on growth options

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do not. This feature of our model is novel and allows us to reproduce both the equity premium and the value premium observed in the data, while maintaining the appealing features of the traditional RBC models on the quantity side.

Our analysis contributes to several strands of literature. We follow Hansen et al. (2005) and Li (2009) and interpret the spread in the return on book-to-market-sorted portfolios as evidence for the difference in the risk premiums of tangible and intangible capital. Hansen et al. (2005) believe that this observation "has potentially important ramifications for how to build explicit economic models to use in constructing measures of the intangible capital stock." The purpose of our paper is to develop such a model and provide a unified framework to both measure and price intangible assets.

Our paper is related to the literature on real options and the cross-section of equity returns (see, e.g., Berk et al. (1999), Gomes et al. (2003), Carlson et al. (2004), Cooper (2006)) and the literature on adjustment costs and value premium (Zhang (2005), Gala (2005)). Our study differs from the above literature along several dimensions, however. First, in our economy, growth options are less risky than assets in place, whereas in previous real options–based models the opposite is true. The aforementioned papers explain the observed value premium by postulating that value firms are option intensive while growth firms are assets in place intensive. Empirical evidence, however, suggests that growth firms are option intensive. Typically, growth firms have higher R&D investment (Li and Liu (2010)) and a higher capital-expenditure-to-sales ratio (Da et al. (2012)), two commonly used empirical proxies for firms' growth opportunities. Growth firms also feature longer cash-flow duration than value firms (see, e.g., Dechow et al. (2004), Da (2006), and Santos and Veronesi (2010)), consistent with the interpretation that their assets consist mainly of options rather than installed physical capital. More recently, Kogan and Papanikolaou (2009) and Kogan and Papanikolaou (2010) provide direct empirical evidence for the lower average return of growth options relative to assets in place. Our framework is consistent with the above empirical findings, since in our economy assets in place have both higher returns and shorter duration than growth

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options.

Second, we work in general equilibrium and study the quantitative implications of our model for asset prices as well as the joint dynamics of consumption, investment, and hours worked. Many of the above papers, however, present partial equilibrium models. Although Gomes et al. (2003) and Gala (2005) adopt a general equilibrium approach, they do not focus on standard RBC moments. In contrast, we use the empirical evidence on the quantity side of the economy to discipline our model of production technology and, therefore, its asset pricing implications. Our unified neoclassical framework combines the success of the RBC models on the quantity side with the success of long-run risk–based models on the cross-section of equity returns obtained in endowment economies.

Third, our model assumes a long-run component in productivity and endogenously generates a long-run component in consumption growth. We show that value stocks are more exposed to long-run shocks than are growth assets. This feature of our model is consistent with the empirical evidence presented in Bansal et al. (2005), Hansen et al. (2008), and Kiku (2006).

Similarly to our approach, Ai and Kiku (2009) also explore conditions under which growth options are less risky than assets in place because of lower exposure to long-run risk. Their analysis differs from ours, however, in that in their model the creation of intangible assets is exogenous, and they do not confront the model with empirical evidence on macroeconomic quantities such as investment or hours worked.

Our paper builds on the literature on asset pricing in production economies, which was recently surveyed by Kogan and Papanikolaou (2011). Our work differs from previous papers in two significant respects. First, our model addresses the equity premium puzzle, as does the rest of the literature, but more importantly we also study the spread between the returns on tangible and intangible capital. Second, this literature typically relies on capital adjustment costs or other frictions in investment to generate variations in the price of physical capital. However, strong adjustment costs, although necessary to generate a sizeable equity premium, are often associated with either a counterfactually low volatility of investment or a counterfactually high volatility of the risk-free

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interest rate. Our model simultaneously produces a low volatility of the risk-free interest rate, a significant volatility of stock market returns, and a high volatility of investment, as in the data.

In a recent study, Borovicka et al. (2011) develop methods to analyze the sensitivity of quantities and asset prices with respect to macroeconomic shocks in dynamic stochastic general equilibrium models. Borovicka and Hansen (2011) focus on the discrete time case and examine the shock-exposure and shock-price elasticities of tangible and intangible capital generated by our model.

Finally, our paper also relates to the literature that emphasizes the importance of intangible capital in understanding macroeconomic quantity dynamics and asset prices. Hall (2001) infers the quantity of intangible capital in the US economy from a capital adjustment cost model. McGrattan and Prescott (2009a, 2009b) emphasize the importance of intangible capital in understanding economic fluctuations. Jovanovic (2008) models intangible capital as investment options and investigates its implications on aggregate Tobin's Q. Gourio and Rudanko (2010) focus on the relationship between customer capital, investment, and aggregate Tobin's Q. Lin (2009) studies intangible capital and stock returns in a partial equilibrium model with capital adjustment cost. Eisfeldt and Papanikolaou (2009) analyze organization capital and the cross-section of expected returns. While providing insights on intangible capital, these papers do not study the difference in the expected return of value and growth stocks.

The remainder of the paper is organized as follows. We present the model and some analytical results in Sections I and II. In Section III, we provide empirical evidence on the lower risk exposure of new investments relative to physical capital of older vintages. We discuss the quantitative implications of our benchmark model in Section IV and consider relevant extensions in Section V. Section VI concludes. Proofs of the theorems and the robustness analysis of the empirical results can be found in the Appendix.

I   Model Setup

A   Preferences

Time is discrete and infinite, $t = 1, 2, 3, \cdots$. The representative agent has Kreps and Porteus (1978) preferences, as in Epstein and Zin (1989):

$$V_t = \left\{ (1 - \beta) \, u \left(C_t, N_t \right)^{1 - \frac{1}{\psi}} + \beta \left(E_t \left[V_{t+1}^{1-\gamma} \right] \right)^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}},$$

where C_t and N_t denote, respectively, the total consumption and total hours worked at time t. For simplicity, we assume an inelastic labor supply and set $u \left(C_t, N_t \right) = C_t$. We relax this assumption in section V.

B   Production Technology

Production Units.   Consumption goods are produced by production units of overlapping generations. Production units created at time τ are called generation-τ production units and begin operation at time $\tau + 1$. Each generation-τ production unit uses labor, n_t^τ, as the only input of production and pays a competitive real wage w_t. For $t \geq \tau + 1$, let A_t^τ denote the time t labor productivity level common to all the production units belonging to generation τ. The output of a generation-τ production unit at time t, y_t^τ, is given by

$$y_t^\tau = \left(A_t^\tau n_t^\tau \right)^{1-\alpha}, \quad \forall t \geq \tau + 1.$$

At the equilibrium, the cash flow of a generation-τ production unit at time t is given by

$$\pi_t^\tau = \max_n \left\{ \left(A_t^\tau n \right)^{1-\alpha} - w_t n \right\}.$$

In our setup, labor productivity, A_t^τ, is generation-specific and captures the heterogenous expo-

sure of production units of different vintages to aggregate productivity shocks. The productivity processes are specified as follows. First, we assume that the log growth rate of the productivity process for the initial generation of production units, Δa_{t+1}, is given by

$$\log \frac{A_{t+1}}{A_t} \equiv \Delta a_{t+1} = \mu + x_t + \sigma_a \varepsilon_{a,t+1}, \tag{1}$$

$$x_{t+1} = \rho x_t + \sigma_x \varepsilon_{x,t+1},$$

$$\begin{bmatrix} \varepsilon_{a,t+1} \\ \varepsilon_{x,t+1} \end{bmatrix} \sim i.i.d.N \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right), \quad t = 0, 1, 2, \cdots.$$

This specification follows Croce (2008) and captures long-run productivity risks.

Second, we impose that the growth rate of the productivity of production units of age $j = 0, 1, ..., t-1$ is given by

$$\frac{A_{t+1}^{t-j}}{A_t^{t-j}} = e^{\mu + \phi_j(\Delta a_{t+1} - \mu)}. \tag{2}$$

Under the above specification, production units of all generations have the same unconditional expected growth rate. We also set $A_t^t = A_t$ to ensure that new production units are on average as productive as older ones.[2] Heterogeneity hence is driven solely by differences in exposure to aggregate productivity risk, ϕ_j. Our empirical investigation in Section III suggests that ϕ_j is increasing in j, i.e., older production units are more exposed to aggregate productivity shocks than younger ones.[3] To capture this empirical fact, we adopt a parsimonious specification of the ϕ_j function as follows:

$$\phi_j = \begin{cases} 0 & j = 0 \\ 1 & j = 1, 2, ... \end{cases}$$

That is, new production units are not exposed to aggregate productivity shocks in the initial period

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[2]Generation-t production units are not active until period $t+1$; therefore, the level of A_t^t does not affect the total production of the economy in period t.

[3]In the data, the productivity process of young firms has a higher idiosyncratic volatility than that of older firms. To capture this fact, generation-specific shocks should be included in equation (2). After solving the model with these additional shocks, however, we find only negligible differences in our results. We therefore choose not to include this additional source of shocks for parsimony.

of their life, and afterwards their exposure to aggregate productivity shocks is identical to that of all other existing generations.

We discuss the empirical evidence on heterogeneous exposure in Section III, and we consider more general specifications of the ϕ_j function in Section V. Providing a microeconomic foundation for this feature of the model is beyond the scope of this study. However, we note that both our empirical evidence and the specification of ϕ_j are consistent with the learning model of Pastor and Veronesi (2009). In their economy, young firms are subject to substantial idiosyncratic risks but have very little exposure to aggregate shocks. The reason is that young firms are embedded with new technologies, which are highly uncertain. It is not optimal to operate these new technologies on a large scale until the uncertainty is reduced with learning. As a result, shocks to young firms have little impact on aggregate quantities. Over time, as young firms age, their productivity becomes more correlated with aggregate output because their technologies are adopted on a larger scale.

In our economy, it is convenient to measure production units of all generations in terms of their generation-0 equivalents. As we show in Appendix A, our specification of the productivity process implies that the output and cash flow of a generation-t production unit are ϖ_{t+1} times greater than those of a generation-0, where

$$\varpi_{t+1} = \left(\frac{A_{t+1}^t}{A_{t+1}} \right)^{\frac{1-\alpha}{\alpha}} = e^{-\frac{1-\alpha}{\alpha}(x_t + \sigma_a \varepsilon_{a,t+1})} \quad \forall t. \tag{3}$$

We use $p_{K,t+1}$ to denote the cum-dividend value of a generation-0 production unit at time $t+1$. Because the cash flow of generation-t production units is ϖ_{t+1} times that of a generation-0 production unit, the value of a new production unit created at time t measured in time-t consumption numeraire is $E_t [\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1}]$, where $\Lambda_{t,t+1}$ denotes the stochastic discount factor. We also assume that a production unit dies with probability δ_K at the end of each period, and death shocks are *i.i.d.* across production units and over time.

Blueprints. The only way to construct a new production unit in this economy is to implement a blueprint. Implementing a blueprint at time t costs $\frac{1}{\theta_t}$ units of physical investment good. We call θ the quality of a blueprint, because blueprints with high θ are more efficient in constructing production units. We allow θ_t to differ across blueprints and evolve stochastically over time to capture idiosyncratic shocks to the profitability of blueprints. At the beginning of each period t, first the value of θ_t is revealed, and then the owner of the blueprint makes the decision whether or not to implement it. A blueprint can only be implemented once, and the implementation choice is irreversible. If not implemented immediately, a blueprint dies with probability δ_S at the end of the period, and death shocks are *i.i.d.* across blueprints and over time.

In our setup, at any time t, the owner of a blueprint faces an optimal stopping problem. She can choose to build a production unit immediately at cost $\frac{1}{\theta_t}$. Alternatively, she may delay the implementation decision into the future. If we denote the value of a blueprint with quality θ_t at time t as $p_{S,t}(\theta_t)$, then the following recursive relation holds:

$$ p_{S,t}(\theta_t) = \max \left\{ E_t \left[\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1} \right] - \frac{1}{\theta_t}, \quad (1 - \delta_S) E_t \left[\Lambda_{t,t+1} p_{S,t+1}(\theta_{t+1}) \right] \right\}. \qquad (4) $$

The first term in the brackets is the payoff of immediate option exercise: implementing a blueprint with quality θ_t at time t costs $\frac{1}{\theta_t}$ amount of general output and creates a generation-t production unit whose value is $E_t \left[\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1} \right]$. The second term is the payoff associated with delaying option exercise: with probability $1 - \delta_S$ the blueprint survives to the next period and obtains another draw of θ_{t+1}.

In our economy, the supply of blueprints is endogenous. At time t, a total measure J_t of new blueprints can be produced by investing J_t units of output. Blueprints created at time t can be used to build production units starting from period $t + 1$.

Interpretation. Production units are the building blocks of assets in place. Their creation requires physical output and their value is reflected in the accounting books. They produce final

goods directly and generate payoffs immediately.

Blueprints are growth options. They capture key features of innovations and new investment opportunities. They are subject to substantial idiosyncratic risk (θ) and are implemented only if their quality becomes high enough. Blueprints do not produce any consumption goods immediately; they only start to do so after being implemented. They are intangible in the sense that they are claims to future output and lack physical embodiment. According to US accounting rules, the cost of developing new blueprints such as innovations and new investment opportunities is typically expensed rather than capitalized. For this reason, we think of J_t as intangible investment. In the rest of the paper, we use the terms blueprints and growth options, and the terms production units and assets in place, interchangeably.

Both production units and blueprints constitute a form of capital, because they can be stored and thus allow investors to trade off current-period consumption against future consumption. Specifically, production units are tangible capital, and blueprints are intangible capital. We are interested in understanding how the different roles played by tangibles and intangibles in aggregate production determine their expected returns.

In our setup, stocks feature high book-to-market ratios (value stocks) if they consist mainly of claims to tangible capital. Conversely, low book-to-market ratio stocks (growth stocks) are intangible capital intensive. At the equilibrium, value premium reflects the difference in the expected returns on tangible and intangible capital.

Our notions of value and growth are also consistent with the empirical evidence on the negative relation between cash-flow duration and book-to-market value (see, e.g., Dechow et al. (2004) and Da (2006)). Our value stocks feature short cash-flow duration because they are mainly claims to assets in place that pay off immediately. Growth stocks, in our model, are long-duration assets, because they load heavily on growth options, which generate cash flows only in the distant future after they are implemented and become production units.

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C Aggregation

Tangible Capital. We use M_t to denote the total measure of production units created at time t and use K_t to denote the productivity-adjusted total measure of production units expressed in generation-0 equivalents. The advantage of using K_t as a state variable is that the aggregate production function is of the Cobb-Douglas form despite the heterogeneity across vintages. If we let Y_t denote aggregate output, then the following holds:

$$Y_t = \sum_{\tau=0}^{t-1}(1 - \delta_K)^{t-\tau-1} M_\tau y_t^\tau = K_t^\alpha \left(A_t N_t\right)^{1-\alpha}, \tag{5}$$

where A_t is the labor productivity of generation-0 production units. In Appendix A, we show that the law of motion of the productivity-adjusted measure of tangible capital, K_t, takes the following simple form:

$$K_1 = M_0, \quad K_{t+1} = (1 - \delta_K) K_t + \varpi_{t+1} M_t, \quad t = 1, 2, \cdots.$$

Our specification of productivity has two advantages. First, it provides a parsimonious way to incorporate the empirical fact that new investments are less exposed to aggregate productivity shocks than capital of older vintages. Second, it maintains tractability at the aggregate level.

Intangible Capital. To avoid having to keep track of the distribution of θ_t as an infinite dimensional state variable, we assume that θ_t is *i.i.d.* among blueprints and over time. For simplicity, we also assume that the distribution of θ_t has a continuous density, denoted as f. As shown in Ai (2009), in this case the mass of newly created production units, M_t, depends only on the total measure of all available blueprints at time t, denoted as S_t, and the total amount of tangible investment goods, I_t, through the following relation:

$$M_t = G\left(I_t, S_t\right) = \max_{\theta_t^*} \left\{ S_t \times \int_{\theta_t^*}^{\infty} f\left(\theta\right) d\theta \right\} \tag{6}$$

$$\text{subject to} \quad S_t \times \int_{\theta_t^*}^{\infty} \frac{1}{\theta} f\left(\theta\right) d\theta \leq I_t,$$

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where the function G is defined as the value function of the optimization problem (6).

Intuitively, optimal option exercise follows a simple cut-off rule: blueprints are implemented in period t if and only if their quality exceeds θ_t^*. Ai (2009) provides a formal proof of this claim and shows that

$$\theta_t^* = G_I(I_t, S_t). \tag{7}$$

In each period, the agent chooses tangible investment, I_t, and exercises top-quality options until the exhaustion of all physical investment goods. Therefore, given the resource constraint in equations (6) and (7), both M_t and θ_t^* are fully determined by I_t and S_t.

Note that one blueprint transforms into exactly one production unit after implementation. Therefore, $G(I_t, S_t)$ is the total measure of both the newly created production units and the blueprints implemented. Taking into account the amount of new blueprints created, J_t, the dynamics of the intangible stock, S_t, is

$$S_{t+1} = [S_t - G(I_t, S_t)](1 - \delta_S) + J_t. \tag{8}$$

Using equation (6), the law of motion of K_t can be written as

$$K_{t+1} = (1 - \delta_K) K_t + \varpi_{t+1} G(I_t, S_t). \tag{9}$$

Finally, we assume that general output can be transformed frictionlessly into consumption, C_t, tangible investment, I_t, and intangible investment goods, J_t, so that the implied aggregate resource constraint is given by

$$C_t + I_t + J_t \leq K_t^\alpha (A_t N_t)^{1-\alpha}. \tag{10}$$

D   Relation to the Literature

Our model of growth options follows the general equilibrium setup in Ai (2009) and differs from existing studies in several respects. First, unexercised growth options can be stored and potentially

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implemented in the future. The storability of unexercised growth options makes them a type of capital distinct from physical assets in place. In contrast, Berk et al. (1999) and Gomes et al. (2003) assume that options disappear if not immediately exercised.

Second, the creation of new growth options in our model is endogenously determined by the optimal choice of the agent. This allows not only the price but also the quantity of intangible capital to adjust to productivity shocks in general equilibrium. The endogenous quantity channel increases the representative agent's ability to smooth consumption and allows options to be *less* risky than assets in place. In contrast, partial equilibrium–based real-option models (for example, Berk et al. (1999), Gomes et al. (2003), and Carlson et al. (2004)) typically assume exogenous arrival of growth options and abstract from the quantity adjustment channel. As a result, options are *more* risky than assets in place in these models.

Third, our intangible capital is the stock of growth options and does not immediately produce output, as does tangible capital. This feature links the cross-sectional differences in both stock returns and their cash-flow duration to production technology. The macroeconomic literature that focuses on the quantity dynamics of intangible capital, in contrast, typically assumes that both intangible and tangible capital affect output directly. For example, the aggregate production function in McGrattan and Prescott (2009a, 2009b) and Corrado et al. (2006) are of the form $Y_t = F(A_t, K_t, S_t, N_t)$, where K_t and S_t denote tangible and intangible capital, respectively. This specification implies that the payments to tangible and intangible capital have similar duration and are both perfectly conditionally correlated with aggregate productivity shocks, thus allowing little room for differences in expected returns.

Finally, the incorporation of intangible capital presents additional challenges to general equilibrium asset pricing models with production. Because of the well-known difficulty in generating a high equity premium in production economies, one might be tempted to assume that intangible capital is much riskier than physical capital and propose this as a resolution of the equity premium puzzle. However, as argued by Hansen et al. (2005), the empirical evidence on the value premium

suggests the exact opposite. In the US, the portfolios of firms with low book-to-market ratios pay substantially lower returns than those of firms with high book-to-market ratios. This suggests that intangible capital earns a much lower risk premium than tangible capital, making it even harder to account for the overall market equity premium. We turn now to the solution of the model and discuss a mechanism that simultaneously generates a high equity premium and a high value premium.

II   Model Solution

A   The Social Planner's Problem

We consider a competitive equilibrium with complete markets in which claims to production units and blueprints are traded. The equilibrium allocation and prices can be constructed from the solution to the social planner's problem that maximizes the representative agent's utility:

$$
V\left(K_t, S_t, x_t, A_t\right) = \max_{C_t, I_t, J_t \geq 0} \left\{ (1-\beta)\, C_t^{1-\frac{1}{\psi}} + \beta \left(E\left[V\left(K_{t+1}, S_{t+1}, x_{t+1}, A_{t+1}\right)^{1-\gamma} \Big| x_t, A_t \right] \right)^{\frac{1-1/\psi}{1-\gamma}} \right\}^{\frac{1}{1-1/\psi}},
$$

subject to the evolution of productivity (equations (1) and (2)), the resource constraint (equation (10)), and the laws of motion of S_t and K_t (equations (8) and (9)). We refer the reader to Ai (2009) for a formal proof of the equivalence between the competitive equilibrium allocation and Pareto optimality.

Despite the heterogeneity in productivity of production units and quality of blueprints, our formulation of the social planner's problem does not use cross-sectional distributions. Our model hence maintains the tractability of standard RBC models—relevant to study macroeconomic quantity dynamics—and simultaneously allows us to the study of both option-exercise and the cross-section of physical and intangible capital returns in general equilibrium.

B Asset Prices

Given equilibrium allocations, the stochastic discount factor of the economy, $\Lambda_{t,t+1}$, can be represented by the ratio of marginal utilities at time t and $t+1$:

$$\Lambda_{t,t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-\frac{1}{\psi}} \left[\frac{V_{t+1}}{\left(E_t \left[V_{t+1}^{1-\gamma} \right] \right)^{\frac{1}{1-\gamma}}} \right]^{\frac{1}{\psi}-\gamma}. \tag{11}$$

Let $p_{K,t}$ and $q_{K,t}$ denote the time-t cum- and ex-dividend price of a generation-0 production unit, respectively. Let $p_{S,t}(\theta)$ denote the value of a blueprint with quality θ at time t before the option exercise decision is made. Because shocks to θ are *i.i.d.* over time, the time-t price of an ex ante identical blueprint before the revelation of θ_t, denoted $p_{S,t}$, is

$$p_{S,t} = \int_0^\infty p_{S,t}(\theta) f(\theta) d\theta,$$

and can be interpreted as the per-unit value of the perfectly diversified aggregate stock of blueprints. We also use $q_{S,t}$ to denote the price of a newly created blueprint at time t.

We use the first-order and envelope conditions of the social planner's problem to characterize the price of growth options and assets in place, as stated in the following proposition.

Proposition 1 *(Equilibrium Conditions) Assets in place are priced as follows:*

$$
\begin{aligned}
p_{K,t} &= \alpha K_t^{\alpha-1} (A_t N_t)^{1-\alpha} + (1-\delta_K) q_{K,t}, \tag{12} \\
q_{K,t} &= E_t [\Lambda_{t,t+1} p_{K,t+1}].
\end{aligned}
$$

A blueprint with quality θ is implemented at time t if and only if $\theta \geq \theta_t^*$, where θ_t^* satisfies

$$E_t [\Lambda_{t,t+1} \varpi_{t+1} p_{K,t+1}] - \frac{1}{\theta^*(t)} = (1-\delta_S) E_t [\Lambda_{t,t+1} p_{S,t+1}]. \tag{13}$$

The price of growth options is determined as follows:

$$p_{S,t} = \frac{G_S(I_{t+1}, S_{t+1})}{G_I(I_{t+1}, S_{t+1})} + (1 - \delta_S) q_{S,t} \tag{14}$$

$$q_{S,t} = E_t[\Lambda_{t,t+1} p_{S,t+1}(\theta_{t+1})] = 1.$$

Proof. See Ai (2009). ■

The two equations in (12) together constitute a recursive relation that can be used to solve for $p_{K,t}$ given equilibrium quantities. The interpretation is that the value of a unit of tangible capital is equal to the present value of its marginal product.

Since a blueprint is implemented at time t if and only if its quality exceeds the threshold level, θ_t^*, equation (13) implies that the owner of a marginal blueprint with quality θ_t^* must be indifferent between immediate option exercise and delaying implementation into the future.

Equation (14) provides a decomposition of option value into in-the-money and out-of-the-money payoff components. The value of an unexercised option is $(1 - \delta_S) q_{S,t}$ after accounting for the death shock. The term $\frac{G_S(I_t, S_t)}{G_I(I_t, S_t)}$ can be interpreted as the expected payoff of an in-the-money option and is an increasing function of $\frac{I}{S}$ by the homogeneity and the concavity of G. Intuitively, a rise in $\frac{I}{S}$ increases the probability of growth options to be exercised and therefore their payoff rises as well. From the social planner's perspective, $\frac{G_S(I,S)}{G_I(I,S)}$ can be interpreted as the marginal product of intangible capital: $G_S(I,S)$ is the number of new production units that can be produced by an additional growth option, and $G_I(I,S)^{-1} = \frac{1}{\theta^*}$ is the price of a marginal production unit measured in current-period consumption goods. The value of an unexercised growth option, q_S, is always 1 because one unit of general output can always be transformed into one unit of new blueprints at time t.

Finally, note that equations (7)–(14) completely characterize both aggregate quantities and prices in the economy. Given aggregate quantities, equation (7) can be used to solve for the optimal option-exercise threshold for blueprints.

The returns of tangible and intangible capital can be therefore written as

$$r_{K,t+1} = \frac{p_{K,t+1}}{q_{K,t}} = \frac{\alpha K_{t+1}^{\alpha-1}(A_{t+1} N_{t+1})^{1-\alpha} + (1 - \delta_K) q_{K,t+1}}{q_{K,t}}, \tag{15}$$

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and

$$r_{S,t+1} \quad = \quad \frac{p_{S,t+1}}{q_{S,t}} = \frac{G_S\left(I_{t+1}, S_{t+1}\right)}{G_I\left(I_{t+1}, S_{t+1}\right)} + \left(1 - \delta_S\right),$$ (16)

respectively. Equations (15) and (16) are the key to understanding the expected returns on tangible and intangible capital. Equation (15) implies, as is common in standard RBC models, that the return on assets in place is monotonic in aggregate productivity shocks. In contrast, the return on growth options does not depend directly on productivity shocks, and in fact it is a function only of the I_{t+1}/S_{t+1} ratio. The return on intangibles is high in states in which the demand for options is large, i.e., when I is large relative to the total supply of growth options, S. Our choice to model intangibles as growth options thus allows the return on physical and intangible capital to depend on different risk factors and, consequently, to command different risk premiums in equilibrium. In Section IV, we show that physical investment I is not responsive to long-run productivity shocks. As a result, the return on intangible capital has little exposure to long-run risk, whereas physical capital is highly risky.

III Firms' Exposure to Aggregate Risks

In this section we provide empirical evidence supporting the claim that new production units are less sensitive to aggregate productivity shocks than are older vintages of physical capital. A production unit in our model should be interpreted as any investment project generating cash flows. Because it is difficult to identify both productivity and age of individual projects within firms, we adopt an indirect approach and work with firm-level data. Specifically, for each firm in our data set we estimate the time-series of its productivity growth rate and compute two alternative measures of the age of its assets in place. We find that the correlation between firm-level and aggregate productivity growth is statistically smaller for firms with younger vintages of physical capital.

A Data and Firm-level Productivity Estimation

Data Description. We consider publicly traded companies on US stock exchanges listed in both the COMPUSTAT and CRSP databases for the period 1950–2008. In what follows, we report COMPUSTAT items in parentheses and define industry at the level of two-digit SIC codes. The output, or value added, of

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firm i in industry j at time t, $y_{i,j,t}$, is calculated as sales (`sales`) minus the cost of goods sold (`cogs`) and is deflated by the aggregate GDP deflator from the US National Income and Product Accounts (NIPA). We measure the capital stock of the firm, $k_{i,j,t}$, as the total book value of assets (`at`) minus current assets (`act`). This allows us to exclude cash and other liquid assets that may not be appropriate components of physical capital. We use the number of employees in a firm (`emp`) to proxy for its labor input, $n_{i,j,t}$, because data for total hours worked are not available.

We construct two measures of the age of assets in place of firm i at time t. Our first measure is simply the age of firms, calculated using founding years from Ritter and Loughran (2004) and Jovanovic and Rousseau (2001). This procedure enables us to form a large dataset with 8,084 different firms and 83,089 observations.

Our second measure is capital age, $KAge_{i,t}$, which we compute as follows:

$$KAge_{i,t} = \frac{\sum_{l=1}^{T}(1 - \delta_i)^l \cdot I_{i,t-l} \cdot l}{\sum_{l=1}^{T}(1 - \delta_i)^l \cdot I_{i,t-l}}, \tag{17}$$

where $I_{i,t}$ measures capital expenditures (`capx`), and δ_i is the firm-specific depreciation rate (depreciation expense (`xdp`) divided by book value of property, plant, and equipment (`ppent`)) averaged over time. When data on depreciation expense are not available, we measure depreciation by COMPUSTAT depreciation (`dp`) minus amortization of intangibles (`am`). According to the above definition, the capital age of a firm is the weighted average age of its capital vintage if we set $T = \infty$. Empirically, we can only choose a finite T and face the following trade-off: a large T provides a better approximation of the age of capital vintage, but it considerably reduces the number of observations in our data set.

In Table 1, we sort all observations in our panel into four firm-age quantiles and present summary statistics. For each quantile, we report median firm age (column 2) and median capital age calculated using $T = 5$, $T = 8$, and $T = 15$ (column 3-5, respectively). All measures of capital age are increasing in firm age, indicating that they are consistent with each other.

Table 1 explicitly shows the trade-off related to the choice of T. If we use the average annual depreciation rate from COMPUSTAT of 15%, setting $T = 15$ implies that we account for roughly 92% of the firms' total capital stock. This choice of T provides a fairly good approximation of the true capital vintage of the firms, but it only allows us to compute capital age for 36% of the 8,084 firms for which firm age is available. On the other hand, setting $T = 5$ permits us to retain all our firms, but this captures only 62% of firms' most recent capital stock. To keep our discussion focused, we present our empirical evidence using firm age as the

Table 1: Summary Statistics by Firm Age Quantiles

Firm Age Quantile	Median Firm Age	Median Capital Age (T=5)	Median Capital Age (T=8)	Median Capital Age (T=15)
1	8	2.55	3.37	4.47
2	16	2.61	3.54	4.84
3	29	2.66	3.63	5.06
4	91	2.71	3.76	5.41
All Firms	24	2.64	3.60	5.01
N. Firms	8,084	8084	6,014	2,937
N. Obs.	83,089	88,283	64,871	32,239

Notes - This table reports the summary statistics of our panel. The sample ranges from 1950 to 2012 and includes approximately 8,084 different firms, for a total of 83,089 observations grouped into four firm-age quantiles. Firm age is expressed in years and is computed using founding dates from Ritter and Loughran (2004) and Jovanovic and Rousseau (2001). Capital age is computed according to equation (17). The last two rows report the number of firms and observations available for different measures of age.

main proxy for the age of firms' production units. In Appendix B, we show that our empirical results are robust to different measures of capital age.

Estimation of Firm-level Productivity. We assume that the production function at the firm level is Cobb-Douglas and allow the parameters of the production function to be industry-specific:

$$y_{i,j,t} = A_{i,j,t} k_{i,j,t}^{\alpha_{1,j}} n_{i,j,t}^{\alpha_{2,j}}, \tag{18}$$

where $A_{i,j,t}$ is the firm-specific productivity level at time t. This is consistent with our original specification since the observed physical capital stock, $k_{i,j,t}$, corresponds to the mass of production units owned by the firm.

We estimate the industry-specific capital share, $\alpha_{1,j}$, and labor share, $\alpha_{2,j}$, using the dynamic error component model adopted in Blundell and Bond (2000) to correct for endogeneity. Details are provided in Appendix B. Given the industry-level estimates for $\widehat{\alpha}_{1,j}$ and $\widehat{\alpha}_{2,j}$, the estimated log productivity of firm i is computed as follows:

$$\ln \widehat{A}_{i,j,t} = \ln y_{i,j,t} - \widehat{\alpha}_{1,j} \cdot \ln k_{i,j,t} - \widehat{\alpha}_{2,j} \cdot \ln n_{i,j,t}.$$

We allow for $\alpha_{1,j} + \alpha_{2,j} \neq 1$, but our results hold also when we impose constant returns to scale in the estimation, i.e., $\alpha_{1,j} + \alpha_{2,j} = 1$.

Table 2: Exposure to Aggregate Risk by Firm Age

Regression	$\Delta \ln \overline{A}$	AGE	$AGE * \Delta \ln \overline{A}$	B/M	Obs.	Firms
(1)	-0.042	-0.002***	0.012***	-0.005	70,909	7,335
	(0.216)	(0.000)	(0.003)	(0.004)		
(2)	0.880*	-0.003***	0.018***	-0.046***	22,432	4,023
	(0.533)	(0.001)	(0.007)	(0.007)		
(3)	0.383	-0.002***	0.011**	-0.006	59,395	7,226
	(0.319)	(0.000)	(0.004)	(0.005)		

Notes - This table reports firms' risk exposure by age. All estimates are based on the following second-stage regression: $\Delta \ln A_{i,j,t} = \xi_{0i} + \xi_1 \Delta \ln \overline{A}_t + \xi_2 AGE_{i,j,t} + \xi_3 AGE_{i,j,t} \cdot \Delta \ln \overline{A}_t + B/B_{i,j,t} + \varepsilon_{i,j,t}$. Regression (1) is obtained using the whole sample. To control for exit bias, in regression (2) we use the Inverse Mills Ratio (IMR) as an additional explanatory variable. In regression (3) we exclude the years with negative aggregate productivity growth. All the estimation details are reported in Appendix B. Numbers in parentheses are standard errors. We use $*$, $**$, and $***$ to indicate p-values smaller than 0.10, 0.05, and 0.01, respectively.

We use the multifactor productivity index for the private nonfarm business sector from the Bureau of Labor Statistics (BLS) as the measure of aggregate productivity.

B   Empirical Results

Here we present our estimates on the link between firm exposure to aggregate productivity and firm age. We provide additional robustness analyses of our results in Appendix B. We consider the following baseline regression:

$$\Delta \ln A_{i,j,t} = \xi_{0i} + \xi_1 \Delta \ln \overline{A}_t + \xi_2 AGE_{i,j,t} + \xi_3 AGE_{i,j,t} \cdot \Delta \ln \overline{A}_t + \xi_4 B/M_{i,j,t} + \varepsilon_{ijt}, \qquad (19)$$

where ξ_{0i} is a firm-specific fixed effect, $\Delta \ln \overline{A}_t$ is the growth rate of aggregate productivity as measured by the BLS, and $B/M_{i,j,t}$ measures firm book-to-market ratio. We introduce the book-to-market ratio to control for the difference in the composition of tangible and intangible assets across firms. The key parameter of interest here is the coefficient ξ_3, which captures the age effect on firm sensitivity to aggregate productivity growth. If the average age of investment projects is increasing in firm age, then under the null of our model ξ_3 is positive.

We find strong empirical evidence in favor of our specification of firm productivity (Table 2). In our baseline estimation (regression (1)), the estimated coefficient ξ_3 is both positive and statistically significant. Furthermore, we obtain very similar point estimates in regressions (2) and (3), where we correct for possible

sample selection bias induced by firm exits.

If exits caused by exposure to negative aggregate productivity shocks are correlated with firm age, they could induce an upward bias in our estimate of ξ_3 in regression (1). Consider a hypothetical scenario in which young firms are more exposed to negative aggregate productivity shocks than are older firms. In such a case, the estimate of ξ_3 obtained from regression (1) would be biased upwards, because young firms would be more likely to exit our database in years with large negative aggregate productivity shocks.

In regression (2) we correct for sample-selection bias by adopting the Heckman (1979) two-stage sample-selection estimator. In regression (3), we instead estimate equation (19) excluding all the observations from years with negative aggregate productivity shocks. The details of these robustness analyses can be found in Appendix B, where we also adopt an additional estimation procedure for the coefficients of the production function. Across all these specifications, our estimates of ξ_3 are very robust: they are consistently positive, statistically significant, and comparable in magnitude.

Note also that the estimate of ξ_4 is consistently negative across all specifications, implying that the productivity growth rate of growth firms is always higher than that of value firms. This is consistent with the view that growth firms have longer cash-flow duration than value firms, a fact that our model replicates and that we address in subsection C.3 of section IV.

Our specification of firms' productivity processes is not only qualitatively consistent with the pattern in the data, but also quantitatively plausible. In fact, our calibration matches well the magnitude of firms' transition from low to high exposure to aggregate productivity shocks. We denote by ϕ_Y (ϕ_O) the regression coefficient of the productivity growth of the young (old) capital vintages on aggregate productivity growth rates. In our model, $\phi_Y = 0$ and $\phi_O = 1.12$. To see why $\phi_O = 1.12$, note that the aggregate productivity growth rate is a weighted average of that of the new capital vintage, $A_{t+1}^t/A_t^t = e^\mu$, and the common growth rate of all older vintages, A_{t+1}/A_t:

$$\Delta \ln \bar{A}_{t+1} = (1 - \lambda_t) \, \Delta \ln A_{t+1} + \lambda_t \mu, \quad 1 > \lambda_t > 0.$$

The regression coefficient of $\Delta \ln A_{t+1}$ on aggregate productivity growth $\Delta \ln \overline{A}_{t+1}$ is therefore $\frac{1}{1-\overline{\lambda}}$. Assuming a death rate of 11% per year for production units, $\overline{\lambda} = 11\%$ in steady state, and $\frac{1}{1-\overline{\lambda}} = 1.12$.

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Table 3: Exposure to Aggregate Risk of Young versus Other Firms

Regression	ϕ				Obs	
	Young	Other	OMY		Young	Other
(1)	-0.484	0.963***	1.447***		15,030	55,879
	(0.373)	(0.100)	(0.361)			
(2)	-0.325	1.202***	1.527*		5,015	17,417
	(0.837)	(0.416)	(0.908)			
(3)	-0.727	1.501***	2.228***		12,721	46,674
	(0.618)	(0.177)	(0.597)			

Notes - This table reports risk exposure of Young and Other firms. In each sample period, a firm is classified as Young if it belongs to the set of the 25% youngest firms; otherwise it is classified in the group Other. All estimates are based on the following second-stage regression (equation (20)):

$$\Delta \ln A_{ijt} = \begin{cases} \xi_{0i} + \phi_Y \Delta \ln \overline{A}_t + \xi_{1i} B/M_{i,j,t} + \widetilde{\varepsilon}_{i,j,t} & i \in \text{Young} \\ \xi_{0i} + \phi_O \Delta \ln \overline{A}_t + \xi_{1i} B/M_{i,j,t} + \widetilde{\varepsilon}_{i,j,t} & \text{otherwise.} \end{cases}$$

OMY refers to $\phi_O - \phi_Y$. Regression (1) is obtained using the whole sample. To control for exit bias, in regression (2) we add the Inverse Mills Ratio (IMR). In regression (3) we exclude the years with negative aggregate productivity growth. All the estimation details are reported in Appendix B. Numbers in parentheses are standard errors. We use $*$, $**$, and $***$ to indicate p-values smaller than 0.10, 0.05, and 0.01, respectively.

In the data, we estimate ϕ_Y and ϕ_O using the following regressions:

$$\Delta \ln A_{i,j,t} = \begin{cases} \xi_{0i} + \phi_Y \Delta \ln \overline{A}_t + \xi_{1i} B/M_{i,j,t} + \widetilde{\varepsilon}_{i,j,t} & i \in \text{Young} \\ \xi_{0i} + \phi_O \Delta \ln \overline{A}_t + \xi_{1i} B/M_{i,j,t} + \widetilde{\varepsilon}_{i,j,t} & \text{otherwise.} \end{cases} \tag{20}$$

In each period a firm is classified as Young if it belongs to the set of the 25% youngest firms in our sample. We report our estimation results in Table 3. Overall, our estimate of ϕ_Y is not statistically different from zero, and that of ϕ_O is positive and significant. The difference in productivity exposure, $OMY = \phi_O - \phi_Y$, is positive and statistically significant, and the point estimate is close to its model counterpart, 1.12.

Our choice of the ϕ_j process is likely to understate the duration of the transition from low to high exposure. Our model assumes that production units have full exposure to aggregate productivity shocks after one period, while the median capital age of the young firms for $T = 15$ is 4.47 years, which suggests that the transition from low exposure to high exposure takes on average 3.47 years. In Section V, we extend our model to allow for more general specifications of the ϕ_j process and show that longer transitions further enhance the equity and value premiums generated by our model. Our current specification for the ϕ_j process

reflects a conservative calibration.

IV Quantitative Implications of the Model

In this section, we calibrate our model at an annual frequency and evaluate its ability to replicate key moments of both macroeconomic quantities and asset returns. We focus on a long sample of US annual data, including pre-World War II data. All macroeconomic variables are real and per capita. Consumption and physical investment data are from the Bureau of Economic Analysis (BEA), while intangible investment (J_t) is measured as in Corrado et al. (2006) by aggregating expenses in brand equity, firm-specific resources, R&D, and computerized information. As in the US NIPA, we treat intangible investment as an expense and define measured output, $Y_{M,t}$, as $C_t + I_t$. Annual data on asset returns are from the Fama-French dataset. We use the Fama-French *HML* factor as a measure of the spread between tangible and intangible capital. Appendix B provides more details on our data sources.

A Parameter Values

Our model has three major components: heterogeneous productivity of vintage capital, long-run productivity risk, and intangible capital. To determine the importance of each component, we compare four different calibrations. The benchmark model comprises all three components and is our preferred calibration. Model 1 lacks heterogeneous productivity of vintage capital (we set $\phi_0 = 1$) but retains the other features of the benchmark model, namely, long-run productivity risk and intangible capital. In model 2, we further exclude fluctuations in long-run productivity growth (by setting $\sigma_x = 0$). Finally, we consider the case without intangible capital in model 3. Essentially, model 3 is the neoclassical growth model with recursive preferences and *i.i.d.* productivity growth rates. The details of the four models are summarized in Table 4.

The parameters of the models can be divided into three groups. The first group includes risk aversion, γ; intertemporal elasticity of substitution, ψ; capital share, α; depreciation rates, δ_K and δ_S; average growth rate of the economy, μ; and the first-order autocorrelation of the predictable component in productivity growth, ρ. These parameters are identical across all four calibrations. We choose the parameters for risk aversion, $\gamma = 10$, and intertemporal elasticity of substitution, $\psi = 2$, in line with the long-run risk literature. We set the capital share $\alpha = 0.3$ and the annual depreciation rate of physical capital $\delta_K = 11\%$, consistent with the RBC literature (Kydland and Prescott (1982)). We choose the same rate of depreciation for

Table 4: Main Components of Our Economy

	Benchmark	Model 1	Model 2	Model 3
Vintage capital	Yes ($\phi_0 = 0$)	No ($\phi_0 = 1$)	No ($\phi_0 = 1$)	No ($\phi_0 = 1$)
Long-run productivity risk	Yes ($\sigma_x \neq 0$)	Yes ($\sigma_x \neq 0$)	No ($\sigma_x = 0$)	No ($\sigma_x = 0$)
Intangible capital	Yes	Yes	Yes	No
Recursive preferences	Yes	Yes	Yes	Yes

Notes - This table summarizes the main components active in each of our four models. All parameter values are reported in Table 5.

intangible capital, $\delta_S = 11\%$. The measured depreciation of intangible capital in our model also includes implemented blueprints, $G\left(I_t, S_t\right)$, and ranges from 40% to 60% per year across the calibrations. Although the empirical evidence on depreciation of intangibles is sparse (Hand and Lev (2003)), these numbers are consistent with the empirical estimate in Corrado et al. (2006). Our sensitivity analysis suggests that δ_S only modestly affects our asset pricing results. We calibrate $\mu = 2\%$ per year, consistent with the average annual real growth rate of the US economy. We set $\rho = 0.93$, which is the point estimate obtained in Croce (2008).

The second group of parameters includes the discount factor, β; the standard deviation of the persistent component of productivity growth, σ_x; and the short-run shock volatility, σ_a. In all calibrations, we set the discount factor β to match the level of the risk-free interest rate in the data if possible. An exception is model 3, which lacks sufficient parameters to match both the level of the risk-free rate and the consumption–tangible investment ratio. We therefore choose β in model 3 to match the consumption–tangible investment ratio but not the level of the risk-free rate. We set σ_a and σ_x in both the benchmark model and model 1 to approximately match the standard deviation and the first-order autocorrelation of the annual growth rate of measured output. In both models 2 and 3, we impose $\sigma_x = 0$ and set σ_a to match the standard deviation of the annual growth rate of measured output.

The third group of parameters describes the functional form of the aggregator $G\left(I_t, S_t\right)$ function. As shown in Ai (2009), for any smooth G function that is concave and homogeneous of degree 1, there is a unique density function $f\left(\theta\right)$ such that G is the aggregator of the option-exercise problem described in equations (4) and (6). We focus our attention on density functions that generate the following CES aggregator:

$$G\left(I, S\right) = \left(\nu I^{1-\frac{1}{\eta}} + \left(1-\nu\right) S^{1-\frac{1}{\eta}}\right)^{\frac{1}{1-1/\eta}} . \tag{21}$$

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Table 5: Calibrated Parameter Values

		Benchmark	Model 1	Model 2	Model 3
Preference parameters					
Discount factor	β	0.97	0.97	0.98	0.89
Risk aversion	γ	10	10	10	10
Intertemporal elasticity of substitution	ψ	2.0	2.0	2.0	2.0
Production function/Aggregator parameters					
Capital share	α	0.3	0.3	0.3	0.3
Depreciation rate of physical capital	δ_K	11%	11%	11%	11%
Depreciation rate of intangible capital	δ_S	11%	11%	11%	–
Weight on physical investment	ν	0.84	0.79	0.815	–
Elasticity of substitution	η	1.40	1.40	1.75	–
Total Factor Productivity parameters					
Average growth rate	μ	2.0%	2.0%	2.0%	2.0%
Volatility of short-run risk	σ_a	5.08%	6.30%	7.30%	5.00%
Volatility of long-run risk	σ_x	0.86%	0.80%	–	–
Autocorrelation of expected growth	ρ	0.925	0.925	–	–
Risk exposure of new investment	ϕ_0	0	1	1	1

Notes - This table reports the parameter values used for our calibrations. The following parameters are common across all models: risk aversion, γ; intertemporal elasticity of substitution, ψ; capital share, α; depreciation rates, δ_K and δ_S; average productivity growth rate, μ. We choose the rest of the parameters to match the moments reported in Table 6 whenever possible. All models are calibrated at an annual frequency.

We choose the two parameters ν and η to approximately match the steady-state consumption–tangible investment ratio and also the consumption–intangible investment ratio across all models, insofar as possible.[4] In Appendix C, we derive the associated density function, f, and the implied cross-sectional distribution of the book-to-market ratio of newly implemented blueprints. We show that our aggregator G, although calibrated to match aggregate moments, conforms well with the microeconomic evidence on the distribution of the book-to-market ratios of new IPO firms in the US.

The calibrated parameter values are summarized in Table 5, and the steady-state moments used to calibrate the parameters are reported in Table 6. We solve the model using a second-order local approx-

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[4]Model 3 does not have intangible capital, so $E[I/J]$ is not defined. In model 2, the parameter η has only minor effects on the stochastic steady state; therefore, it is not possible to match both $E[C/I]$ and $E[I/J]$ simultaneously. In model 2, we follow the RBC literature and set ν to match the consumption–physical investment ratio observed in the data.

Table 6: Moments Used for Model Calibration

	Data	Benchmark	Model 1	Model 2	Model 3
$E\left[C/I\right]$	5.62	5.60	5.54	5.63	5.69
$E\left[I/J\right]$	1.00	1.01	0.95	(0.77)	—
$\sigma\left[\Delta\ln Y_M\right]$	3.49	3.49	3.49	3.49	3.52
$AC1\left[\Delta\ln Y_M\right]$	0.45	0.25	(0.60)	0.46	0.13
$E\left[r_f\right]$	0.86	0.80	0.87	0.86	(12.65)

Notes - This table reports the moments used to calibrate the parameters of the models evaluated in this paper. Our database refers to US annual data from 1930 to 2003 (see Appendix B). All moments that cannot be matched are in parentheses. In Model 1, the autocorrelation of measured output, $Y_M \equiv C + I$, is too high. In Model 2, the parameter ν is set to match the C/I ratio, even though the implied I/J ratio is lower than in the data. In Model 3, the discount factor β is chosen to match the steady-state consumption-investment ratio, even though this choice makes the risk-free interest rate too high.

imation computed using the `dynare++` package. Our results are consistent with those of Borovicka and Hansen (2011), who adopt alternative numerical procedures to analyze shock-price and shock-exposure elasticities generated by our model. We also solve our models numerically using a finite element–based global approximation method to check the accuracy of the local approximation method. Overall, the two numerical solutions produce very similar results.

B Quantity Dynamics

In this section, we show that all four models produce largely similar macroeconomic quantity dynamics and that our benchmark model improves slightly upon the RBC model (model 3) along several dimensions. In this sense, our model inherits the success of the RBC models on the quantity side of the economy.

The quantity dynamics produced by our calibrations are shown in the top panel of Table 7. All four calibrations produce a small volatility of consumption growth and a high volatility of tangible investment growth, consistent with the data. Recall that model 3 is essentially the standard RBC model with recursive preferences. We know from Tallarini (2000) that the risk aversion parameter of the recursive preference has little effect on the quantity dynamics. Therefore, on the quantity side, the model behaves just like the standard RBC model with CRRA preferences where $\gamma = \frac{1}{\psi} = 0.5$. The second moments generated by model 3 are consistent with those in Kydland and Prescott (1982). In particular, this model produces a small standard deviation of consumption (2.14% per year) and a standard deviation of investment about six times larger (15.33% per year).

Moments	Data	Benchmark	Model 1	Model 2	Model 3
$\sigma\left(\Delta\ln C\right)$	02.53 (00.56)	02.60	02.85	02.69	02.14
$\sigma\left(\Delta\ln I\right)$	16.40 (03.24)	14.18	10.90	08.85	15.33
$AC_1\left(\Delta\ln C\right)$	00.49 (00.15)	00.48	00.68	00.48	00.59
$\rho\left(\Delta\ln C, \Delta\ln I\right)$	00.39 (00.29)	00.17	00.56	00.77	00.59
$\rho\left(\Delta\ln C_{10}, \Delta\ln I_{10}\right)$	00.62 (00.24)	00.73	00.82	00.83	00.72
$\sigma\left[SDF\right]$		87.98	90.94	73.50	43.21
$E[r_K - r_f]$		01.95	00.80	00.82	00.31
$\sigma[r_K]$		01.99	01.26	01.27	00.92
$E[r_S - r_f]$		00.54	00.74	00.47	–
$\sigma[r_S]$		00.88	00.88	00.79	–
$\sigma[r_f]$	00.97 (00.31)	01.00	00.96	00.70	00.65
$E[r_M^L - rf]$	05.71 (02.25)	05.20	02.37	02.31	00.83
$E[r_K^L - r_S^L]$	04.32 (01.39)	04.20	00.17	01.05	–

Notes - All figures are multiplied by 100, except contemporaneous correlations (denoted by ρ) and first-order autocorrelations (denoted by AC_1). Empirical moments are computed using US annual data from 1930 to 2003. Numbers in parentheses are GMM Newey-West adjusted standard errors. $\Delta\log C_{10}$ and $\Delta\log I_{10}$ denote the 10-year growth rate of consumption and investment, respectively. $E\left[r_K^L - r_S^L\right]$ measures the average difference between the levered returns of tangible and intangible capital. We use the HML Fama-French factor as an empirical counterpart of $r_K^L - r_S^L$. r_M^L indicates levered market returns. Returns are in log units and the leverage is 3 (Garca-Feijo and Jorgensen (2010)). All the parameters are calibrated as in Table 5. The entries for the models are obtained by repetitions of small-sample simulations.

Comparing models 2 and 3, we see that the addition of intangible capital to the standard RBC model reduces the volatility of physical investment growth. This is because the concavity of the aggregator G implies decreasing marginal production of physical investment and affects the volatility of physical investment similarly to adjustment cost functions in neoclassical models. In order to generate a high volatility of tangible investment, therefore, the curvature of $G\left(I, S\right)$ needs to be low, or, equivalently, the elasticity of substitution between I and S, η, needs to be sufficiently high. All of our calibrated models with intangible capital have this feature. Adding long-run shocks and heterogenous productivity of capital vintages increases the volatility of investment. In model 2, investment growth volatility is almost 11%, and in the benchmark model it reaches a level of 14.18%, consistent with the data.

The persistence of the growth rates of macroeconomic quantities produced by our model is similar to that in the data. In models 2 and 3, both output and consumption are autocorrelated, even if productivity growth is not. This result is generated by the persistent fluctuations of our endogenous state variables, K and S (as in Kaltenbrunner and Lochstoer (2010)). The persistence generated in these two models, however,

is smaller than that in the data. The addition of long-run productivity risk increases the autocorrelation of consumption and output growth rates (Croce (2008)). Since both the benchmark model and model 1 feature long-run productivity shocks, they produce a higher autocorrelation in output growth (Table 6) and consumption growth (Table 7) than models 2 and 3. The introduction of long-run productivity shocks, therefore, brings our model closer to the data.

The correlation between consumption and physical investment growth rates in our benchmark calibration is consistent with its empirical pattern, i.e., it is moderate at an annual frequency and high over the long horizon. This is an improvement with respect to standard RBC models, which are notorious for producing large correlations of consumption and investment growth even over short horizons. Standard RBC models have only one source of shocks, the short-run productivity shocks. Since both consumption and investment co-move with this shock, the correlation of their growth rates is quite high. In contrast, our model also features news about future productivity shocks that have no effect on current total output. Because of the resource constraint, consumption and total investment must move in opposite directions in response to these shocks, reducing their unconditional correlation.

In section V, we show that extensions of our model are capable of matching a broader set of moments, including intangible investment volatility and the dynamics of hours worked.

C   Asset Price Dynamics

In this section we examine the asset pricing implications of our model. While the quantity dynamics of the benchmark model inherits the basic features of the standard RBC model, thanks to the lagged risk exposure of new vintage capital, asset returns in our model respond to long-run risks similarly to endowment-based long-run risk models, for example, that of Bansal and Yaron (2004). More importantly, our model is able to produce a large spread between the expected return on tangible and intangible capital.

Campbell (2000) summarizes the challenge to general equilibrium asset pricing models as three puzzles: the equity premium puzzle (Mehra and Prescott (1985)), the stock market volatility puzzle (Campbell (1999)), and the risk-free rate puzzle (Weil (1989)). These puzzles are even more difficult to solve in production economies, as models must (1) not only generate a pricing kernel that is sufficiently volatile, but also endogenously produce a high risk exposure of the stock market returns, and (2) be consistent with the empirical evidence from the quantity side of the economy. The literature has relied primarily on adjustment cost or other forms of rigidity in investment to generate the variation in the price of physical

capital. In the next sections we show that it is difficult to reconcile the high risk exposure of the market returns and the high volatility of tangible investment when relying on rigidity in investment as the only means by which to generate variations in the price of physical capital. In our benchmark model, however, thanks to heterogeneous productivity of capital of different vintages, we can simultaneously produce volatile stock market returns and aggregate physical investment. The adoption of recursive preferences with high intertemporal elasticity of substitution also allows us to solve the risk-free rate puzzle.

The empirical evidence on the value premium imposes a strong discipline on general equilibrium asset pricing models with intangible capital. Stocks with high book-to-market ratios earn higher returns than stocks with low book-to-market ratios, and the difference between market value and book value can be attributed to the value of intangible capital owned by the firm. This evidence suggests that intangible capital earns a lower average return than physical capital. Qualitatively, the benchmark model and models 1 and 2 are consistent with intangible capital being less risky than physical capital (Ai (2009)). Quantitatively, however, only the benchmark model is capable of producing a significant value premium. The interaction between lagged risk exposure of new vintage capital and long-run productivity risk is the main driver of this result.

In the following subsections, we first discuss the common features of all four calibrations (Section C.1), then examine the models' implications for the returns on physical capital r_K (Section C.2). Finally, we study the models' implications for the value premium (Section C.3). The asset pricing implications of all four calibrations are summarized in Table 7.

C.1   Common Features

All calibrations except model 1 are able to generate a low and relatively smooth risk-free interest rate.[5] The volatilities of the risk-free interest rates are low because we adopt an intertemporal elasticity of substitution greater than one: since agents are very willing to substitute consumption across time, fluctuations in the expected consumption growth rate produce only small variations in the equilibrium interest rate.

All four models produce a fairly high volatility of the stochastic discount factor. Since the representative agent is endowed with recursive preferences, fluctuations in expected consumption growth (long-run risk,

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[5]In standard RBC models, there is always a tension between simultaneously producing a high consumption–physical investment ratio and a low level of the risk-free rate through the subjective discount factor β. This explains why in model 3 we are not able to match the level of the risk-free rate, since we set β to reproduce the consumption-investment ratio observed in the data.
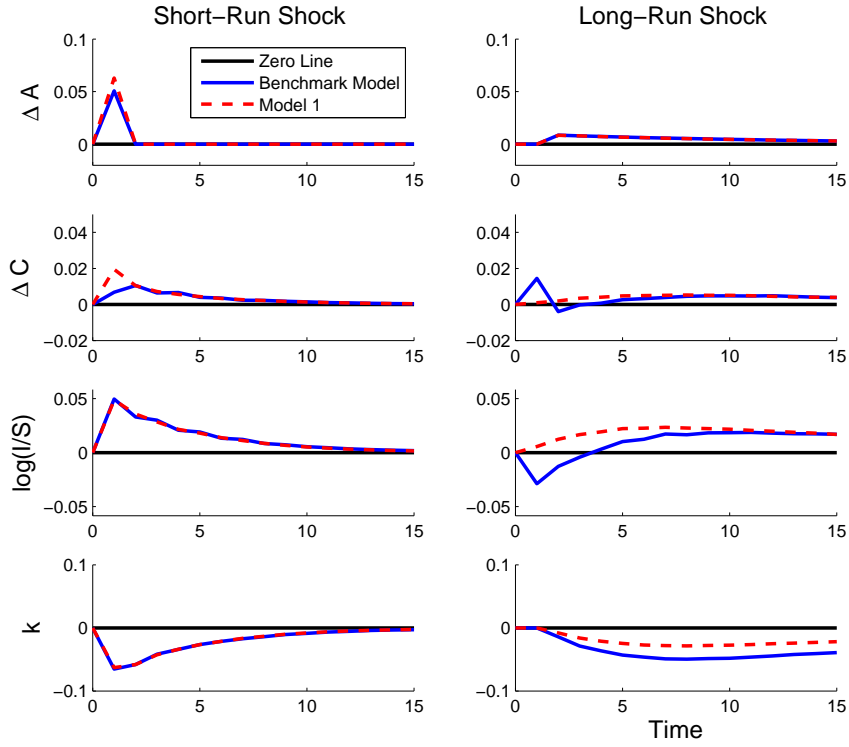
Figure 1 – Impulse Response Functions for Quantities

This figure shows annual log deviations from the steady state. All the parameters are calibrated to the values reported in Table 5. The dashed lines refer to model 1; the solid lines refer to the benchmark model.

in the language of Bansal and Yaron (2004)) strongly affect marginal utility. Models 2 and 3 feature predictability in consumption growth because of the endogenous fluctuations in K and S. The introduction of long-run productivity shocks in both the benchmark model and model 1 almost doubles the volatility of the stochastic discount factor.

C.2 Investment Dynamics and Physical Capital Returns

As shown in Kaltenbrunner and Lochstoer (2010) and Croce (2008), an important challenge for the long-run risk–based asset pricing model with production is to account for the high volatility of investment and stock returns simultaneously. Although recursive preferences generate a high volatility of the stochastic discount factor, the return to physical capital is typically very smooth, unless one is willing to assume a large adjustment cost. High levels of adjustment cost, however, are typically associated with counterfactually low levels of volatility in investment growth.

This tension is present in models 1, 2, and 3, but it is resolved in our benchmark model, where the annual volatility of the unlevered returns on physical capital is 2.00% and investment is as volatile as in an RBC model. To explain our results, we plot in Figures 1 and 2 the impulse response functions of quantities and prices, respectively, generated by both short-run and long-run shocks in the benchmark model and model 1.

The left panels of Figures 1 and 2 show that the introduction of heterogeneous productivity of vintage capitals does not significantly alter the model's response to short-run shocks. This result has two important implications. First, since the quantity dynamics in the benchmark model are mostly driven by short-run shocks, they inherit the success of standard RBC models with *i.i.d.* productivity growth (model 3). Second, the risk premiums associated with short-run shocks are small in both models. Therefore, in order to understand the success of our benchmark model in accounting for both equity and value premiums, we must focus on the interaction between long-run shocks and the heterogeneous productivity of vintage capitals.

As shown in the right-hand panels of Figures 1 and 2, the impulse responses to long-run shocks are significantly different across model 1 and the benchmark model. With a one-standard-deviation change in the long-run productivity shock, the return on physical capital, r_K, in the benchmark model increases by about 1.5%, whereas the change in r_K in model 1 is barely visible. This implies that the exposure to the long-run productivity risk of physical capital is very small in model 1, whereas that in the benchmark model is larger by several orders of magnitude.

To explain the different behavior of r_K across the benchmark model and model 1, we focus our attention on the ex-dividend price of physical capital, $q_{K,t}$ (see Figure 2, fourth panel, right column). Iterating equation (12) forward, we can express $q_{K,t}$ as the present value of the infinite sum of all future payoffs:

$$q_{K,t} = \sum_{j=1}^{\infty} (1 - \delta_K)^j \, E_t \left[\Lambda_{t,t+j} \alpha K_{t+j}^{\alpha-1} \left(A_{t+j} \right)^{1-\alpha} \right], \tag{22}$$

Equation (22) implies that the price of physical capital, $q_{K,t}$, is the present value of the marginal product of physical capital in all future periods. This equation holds in model 1 as well. A positive innovation in the long-run productivity component x_t has two effects on the future marginal product of physical capital. The first is a direct effect: keeping everything else constant, an increase in x_t raises the marginal product of physical capital by increasing all future A_{t+j} for $j = 1, 2, \cdots$. The second effect comes from the general equilibrium. An increase in the marginal productivity of capital also triggers more investment, which augments K_{t+j} in all future periods. Due to the decreasing returns to scale ($\alpha < 1$), an increase in K_{t+j} mitigates the direct

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effect.

In model 1, the elasticity of substitution between physical investment and intangible capital, η, is set to 1.4. This implies that the supply of physical investment is quite elastic. Consequently, the return on physical capital responds very little to long-run shocks. To see this point more clearly, note that without overlapping generations of vintage capital, we have $\varpi_t = 1 \quad \forall t$, and equation (13) can be written as

$$q_{K,t} - (1 - \delta_S) = \frac{1}{G_I(I_t, S_t)} = \frac{1}{\nu} \left(\frac{I_t}{G(I_t, S_t)} \right)^{\frac{1}{\eta}}. \tag{23}$$

By equation (23), as η increases, I_t becomes more sensitive to changes in $q_{K,t}$. Equation (22) implies that if investment adjusts elastically to productivity shocks, then the effect of the long-run productivity shock on $q_{K,t}$ is small, due to decreasing return to scale of physical capital. This intuition is confirmed by our impulse response functions. Innovations in the long-run productivity component are accompanied by a nearly permanent increase in the I/S ratio (Figure 1, third panel, right column, solid line). As a result, the changes in q_K after a long-run productivity shock are almost negligible (Figure 2, fourth panel, right column). To summarize, in model 1 the return on physical capital responds little to long-run productivity shocks because the direct effect on the price of physical capital is mostly offset by movements in investment (the general equilibrium effect). As with standard adjustment cost models, it is difficult to simultaneously produce a high volatility of both investment growth and returns on physical capital in model 1.

In the benchmark model, however, after a long-run productivity shock, investment rises, but after a substantial delay, whereas the return on physical capital increases immediately and sharply. The I/S ratio initially drops and then starts to rise, always staying below the level obtained in model 1 (Figure 1, fourth panel, right column). The last panel in the right column of Figure 1 plots the impulse response of physical capital stock normalized by productivity ($k_t = K_t/A_t$) after a long-run shock. Because of the lagged response of investment, the level of physical capital in the benchmark model stays nearly permanently behind that obtained in model 1. Since the marginal product of capital, $\alpha k_t^{-(1-\alpha)}$, is a decreasing function of normalized capital stock, in the benchmark model the marginal product of physical capital remains almost permanently above that observed in model 1, producing a strong increase in $q_{K,t}$.

In this case, the direct and general equilibrium effects of long-run productivity shocks affect $q_{K,t}$ in the same way, thereby reinforcing each other. The marginal product of capital increases both because a positive shock in x_t increases A_{t+j} in all future periods and because the sluggish response of investment to long-run
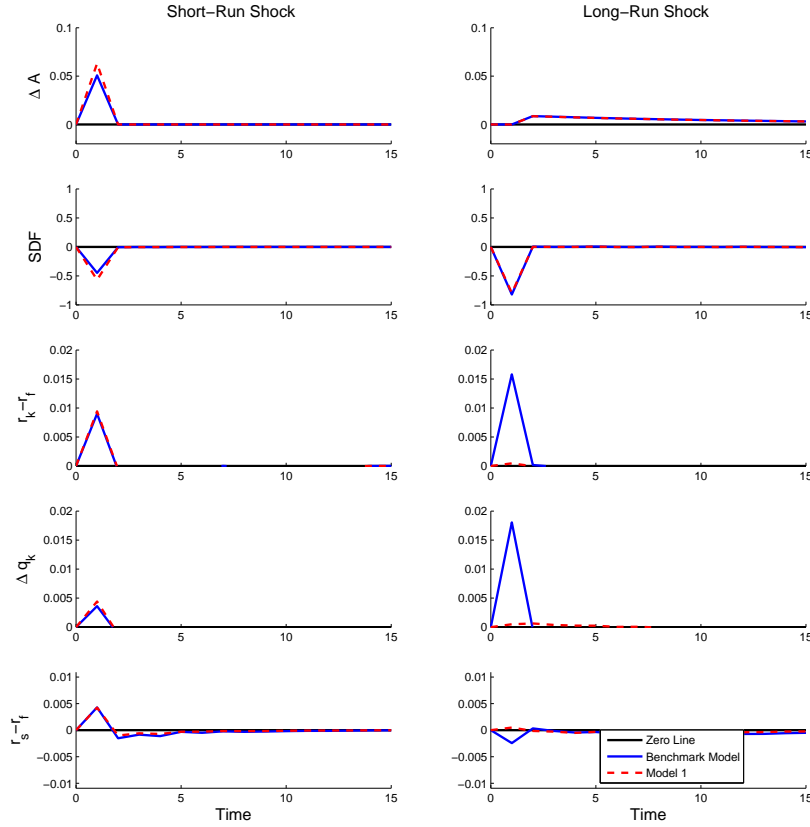
Figure 2 – Impulse Response Functions for Prices

This figure shows annual log deviations from the steady state. All the parameters are calibrated to the values reported in Table 5. Returns are not levered. The dashed lines refer to model 1; the solid lines refer to the benchmark model.

shocks results in a nearly permanent reduction of physical capital stock relative to that in model 1.

To understand the lagged response of investment to long-run news in the benchmark model, note that a long-run shock increases the productivity of all existing vintages of capital almost permanently but affects the productivity of the new production units only after a delay. This generates an incentive to postpone the exercise of new growth options. As a result, a long-run productivity shock immediately produces a strong income effect (the agent anticipates a persistent increase in the productivity of all existing vintages of capital and prefers to consume more) without generating a significant substitution effect (the return on new physical investment is unaffected by long-run productivity shocks for an extended period of time). At time 1, when a

positive long-run shock materializes, the net effect is an immediate increase in consumption and a decrease in investment, exactly the opposite of what happens in model 1, in which the substitution effect dominates the income effect and investment increases. This feature of the model is consistent with recent empirical findings of Barsky and Sims (2010) and Kurmann and Otrok (2010).

In the benchmark model, positive long-run shocks, although small, have quite significant and prolonged negative effects on physical investment. This sluggish response of investment is generated by the persistence of the long-run shocks: after positive long-run news, the relative productivity of new investment remains behind that of existing vintages for an extended period of time, thereby discouraging a fast and full recovery of investment.

C.3   Value Premium

We report the value premium in the data and the model in the last row of Table 7. In the data, *HML* is calculated as the average return of the *HML* factor as constructed by Fama and French (1995). The model counterpart of *HML* is calculated as the difference in the leveraged return on tangible and intangible capital.[6]

To understand the difference in the expected returns of tangible and intangible capital, we can use the functional form of $G(I, S)$ in equation (21) and write the returns of intangible capital in equation (16) as

$$r_{S,t+1} = \frac{1-\nu}{\nu} \left(\frac{I_{t+1}}{S_{t+1}}\right)^{\frac{1}{\eta}} + (1 - \delta_S). \tag{24}$$

As explained in section II.B, the term $\frac{1-\nu}{\nu} \left(\frac{I_{t+1}}{S_{t+1}}\right)^{\frac{1}{\eta}}$ can be interpreted as the expected payoff of in-the-money options in period $t + 1$. Because S_{t+1} is determined in period t, innovations in the return on intangible capital respond positively to innovations in I_{t+1}. The intuition for this result is that an increase in the $\frac{I}{S}$ ratio lowers the option-exercise threshold $\theta^*(t) = G_I(I_t, S_t)$ and raises the probability of option-exercise, thereby enhancing the payoff of growth options. As shown in Figures 1 and 2, in our benchmark model, $\frac{I}{S}$ responds negatively to long-run productivity shocks. Therefore, our model is able to account for the empirical fact that growth stocks are less exposed to long-run economic risks, as documented in Bansal et al. (2005), Hansen et al. (2008), and Kiku (2006).

To understand the lower exposure of growth options with respect to long-run productivity shocks com-

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[6]In our analysis, we abstract away from both financial and operative leverage. Garca-Feijo and Jorgensen (2010) estimates suggest a degree of total leverage of 4; we set leverage to 3 to be conservative.

pared to assets in place, note that the payoff of growth options can be replicated by long positions in assets in place and short positions in the cost of the strike asset. Exercising growth options at time t costs $\frac{1}{\theta}$ unit of investment goods and produces a generation-t production unit, which is equivalent to ϖ_{t+1} generation-0 production units. Since installed physical capital in this economy is measured in terms of generation-0 production unit equivalents, the expected cost of creating an additional unit of K_t on date t is $E_t\left[\varpi_{t+1}^{-1}\frac{1}{\theta}\right] = \frac{1}{\theta}e^{\frac{1-\alpha}{\alpha}\left(x_t+\frac{1}{2}\sigma_a^2\right)}$, $0 < \alpha < 1$. Growth options have low exposure to long-run risk, x_t, because the cost of exercising them, $E_t\left[\varpi_{t+1}^{-1}\frac{1}{\theta}\right]$, covaries positively with x_t and acts as a hedge. Good news for the productivity of existing productions units is bad news for unimplemented blueprints because it is more expensive to create new production units as productive as those of old vintages. As a result, both physical investment and option returns respond negatively to long-run productivity shocks.

The implications of our model for the value premium are summarized in the bottom panel of Table 7. We make the following observations. First, all models with intangible capital yield a higher return for physical capital than for intangible capital. Second, despite the introduction of long-run risk, model 1 produces a lower spread between physical and intangible capital than does model 2. In model 1, intangible capital is more exposed to long-run risk than is tangible capital. Specifically, without heterogeneous productivity of vintage capital, after a positive long-run productivity shock, physical investment increases sharply but $q_{K,t}$ remains almost flat (Figure 2). At the same time, the increase in the I/S ratio is associated with a drop in the option exercise threshold, $\theta^*(t)$, and a positive innovation in the return on intangible capital. As a result, as shown in Table 7, simply adding long-run productivity shocks to model 2 increases the market risk premium only slightly and eliminates most of the spread in the expected return on physical and intangible capital.

Third, compared to model 1, our benchmark model produces both a larger risk premium on physical capital and a smaller one on intangible capital, thus improving on equity and value premiums simultaneously. The heterogeneous productivity of vintage capital is responsible for both improvements because it causes the I/S ratio to drop after good long-run news. This feature of the model produces a sharp increase in the return on tangible capital, r_K, and a drop in the intangible capital return, r_S. It increases the riskiness of physical capital and makes growth options an insurance device against long-run risk. Overall, the benchmark model produces a market risk premium more than two times larger than that of model 1, and a spread between tangible and intangible capital returns larger by an order of magnitude.

We conclude our discussion on value premium by exploring the implications of our model for the cash-

flow duration of book-to-market-sorted portfolios. We define the Macaulay duration, MD_t, of a stochastic cash flow process, CF_t, as:

$$MD_t = \frac{\sum_{s=1}^{\infty} s \cdot E_t \left[\Lambda_{t,t+s} CF_{t+s} \right]}{\sum_{s=1}^{\infty} E_t \left[\Lambda_{t,t+s} CF_{t+s} \right]}. \tag{25}$$

We provide the details of the calculation of the duration of growth options and assets in place in Appendix C. Here we point out that options typically have longer duration than assets in place because they start paying cash flows only after being exercised and becoming assets in place. Under our benchmark calibration, the Macaulay duration of assets in place (17 years) is about half of that of growth options (30 years).

Since in our model value stocks are intensive in assets in place and growth stocks are intensive in options, our framework is consistent with the inverse relationship between cash-flow duration and book-to-market characteristics documented by Dechow et al. (2004). This feature of our model stands in contrast with previous results in the real-option literature. In Gomes et al. (2003), for example, value stocks are option intensive and therefore have longer cash-flow duration than low book-to-market stocks.

D   Additional Testable Implications of the Model

In this section, we conduct econometric analyses on model predictions that directly link asset prices to macroeconomic fundamentals. First, we provide supporting evidence on the response of both investment growth and the spread between tangible and intangible capital returns to productivity news shocks. Second, we study the correlation of investment leads and lags with aggregate stock market returns as well as the spread between tangible and intangible capital returns.

Response to News Shocks.   The key asset pricing implications of our model rely on the exposure of asset returns to long-run risk, or news about future productivity shocks. Our production-based general equilibrium framework links risk exposure to the response of macroeconomic quantities to these shocks. As we discuss in section IV.C, a positive news shock is accompanied by a sharp increase in the spread of the returns of tangible and intangible capital. On the quantity side, it leads to an immediate decrease in aggregate investment and a corresponding rise in aggregate consumption without affecting total output. We test these conditional responses in the data by jointly estimating equation (1) and the following system of

equations through a GMM procedure:

$$x_t = \beta_{rf}^{da} \cdot r_{f,t-1} + \beta_{pd}^{da} \cdot pd_{t-1} \tag{26}$$

$$HML_t = \beta_{SR}^{HML} \cdot \varepsilon_{a,t} + \beta_{LR}^{HML} \cdot \varepsilon_{x,t} + \varepsilon_t^{HML} \tag{27}$$

$$CY_t = \beta_{SR}^{C/Y} \cdot \varepsilon_{a,t} + \beta_{LR}^{C/Y_M} \cdot \varepsilon_{x,t} + \beta_{(-1)}^{C/Y_M} \cdot CY_{t-1} + \varepsilon_t^{C/Y_M}, \tag{28}$$

where $CY_t = \frac{C_t}{C_t + I_t}$ is the consumption-output ratio, or consumption share. In equation (26), we follow Bansal et al. (2007) and use the risk-free rate and the log price-dividend ratio to identify news shocks. Equation (27) is also consistent with our model: the spread between the value and growth portfolios, HML_t, depends on the realization of short-run and long-run productivity shocks, as well as an error term, ε_t^{HML}. In equation (28), we use $\beta_{SR}^{C/Y}$ and $\beta_{LR}^{C/Y}$ to denote the sensitivity of the consumption-output ratio with respect to short-run and long-run productivity shocks, respectively. Consistent with our model, the consumption share process is very persistent in the data. Instead of estimating a full-blown DSGE model with K_t and S_t as state variables, we use the lagged value, CY_{t-1}, to control for the history dependence of the consumption share process.[7]

In Table 8, we report our results for three different measures of aggregate productivity. In the first row, we compute aggregate productivity according to equation (5). We set $\alpha = 0.3$ and assume an inelastic labor supply, $N_t = 1$, as in our benchmark model. In the second regression, we allow for changes in aggregate labor. Data on labor and physical capital are from the NIPA tables and are described in Appendix B. In our third specification, we use a multifactor adjusted measure of productivity directly provided by the BLS. The first two measures of productivity can be computed starting from 1930, but the BLS productivity data are available only for the post–World War II period. Across all specifications, the identification of the long-run predictable component in productivity growth is statistically significant, as shown by the small p-value of the Wald statistics. In addition, both HML and the consumption share respond positively to the identified news shocks, as predicted by our model. The response of HML is positive and statistically significant across all specifications. The response of consumption share, in contrast, is statistically significant in regressions (1) and (2) when longer samples of the productivity data are available.

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[7]We have also estimated a version of equation (28) including investment-specific shocks, an alternative determinant of the consumption-output ratio discussed in the literature. Our results are robust to this extension. We thank Dimitris Papanikolaou for sharing his data on investment specific shocks.

**Table 8: Conditional Responses of Consumption and HML**

	Productivity	Sample	C/Y		HML		Wald Test
			SRR	LRR	SRR	LRR	p-val
(1)	Capital adjusted (BEA)	1930–2006	-0.403*** (0.041)	0.287* (0.158)	5.015 (23.076)	127.361*** (43.259)	0.000
(2)	Capital and labor adjusted (BEA)	1930–2006	0.189*** (0.042)	0.297*** (0.131)	1003.184*** (89.484)	1106.65*** (292.082)	0.000
(3)	Multifactor adjusted (BLS)	1949–2006	-0.397*** (0.052)	0.003 (0.093)	72.81 (59.84)	1713.036*** (213.999)	0.000
(4)	O-Y capital adjusted	1951–2006	0.019 (0.018)	0.174*** (0.037)	-1.478 (9.053)	162.153*** (23.876)	0.000
(5)	O-Y capital and labor adjusted	1951–2006	0.023** (0.011)	0.032 (0.050)	4.745 (12.755)	666.628*** (117.059)	0.000

Notes - This table reports contemporaneous sensitivity of consumption share and HML to both short- and long-run productivity shocks. Specifically, we jointly estimate equations (1) and (26)–(28) and report $\beta_{SR}^{C/Y}$, $\beta_{LR}^{C/Y}$, β_{SR}^{HML}, and β_{LR}^{HML} in columns 3 to 6, respectively. In the last column, we report the p-value of the Wald statistics that tests the null of no predictability in productivity growth. The first three regressions are based on aggregate measures of productivity. The last two regressions are based on the productivity differential between Young and Other firms as defined in section III to proxy for $-\varpi_t$. Numbers in parentheses are GMM standard errors. We use $*$, $**$, and $***$ to indicate p-values smaller than 0.10, 0.05, and 0.01, respectively.

In addition, we test the implications of our model for the responses of returns and quantities to news about the differences in productivity of young and old firms. We construct the productivity difference as the ratio of the average productivity of all firms in the second to fourth age quartiles and that of the youngest 25% of firms in our data set (see section III). We estimate equations (1) and (26)–(28), with Δa_t replaced by the log difference of productivity. Because the productivity difference is not perfectly correlated with aggregate productivity in the data, this estimation exercise provides yet another way to empirically test the model.

We report the results of our estimation in row (4) of Table 8, where firm-level productivity is computed as in equation (18) but assuming an inelastic labor supply. In row (5) of the same table, we allow for variations in firm-level labor as in Section III. Consistent with our model, both HML and the consumption share drop upon the arrival of good news for the relative productivity of young and old firms.

Leads and Lags. The economic mechanism in the benchmark model has strong implications for the correlation of tangible investment with the market return and the spread between tangible and intangible capital returns. We plot these correlations in Figure 3 for both our benchmark model and model 1 and show that our benchmark model fits the correlation patterns in the data well.
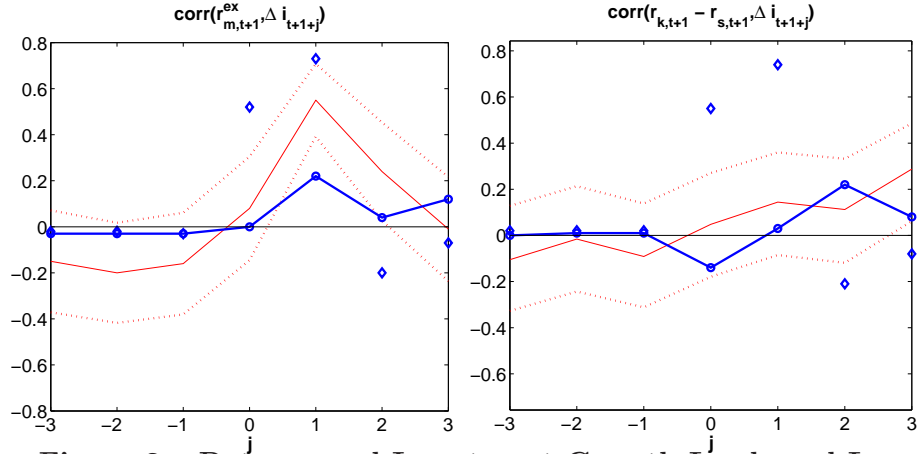
Figure 3 – Returns and Investment Growth Leads and Lags

This figure shows the correlation of market excess returns (left panel) and the spread between the returns of tangible and intangible capital (right panel) with investment growth leads ($j > 0$) and lags ($j < 0$). The thin solid line represents the point estimate of the correlations computed using US data from 1930 to 2003. The spread between tangible and intangible capital is proxied by the HML factor. The dotted lines mark the 95% confidence interval for the correlations. The solid line with circles represents the correlations obtained in the benchmark model. The diamond-shaped markers refer to model 1. All the parameters are calibrated to the values reported in Table 5. The entries from the models are obtained through repetitions of small-sample simulations.

The left panel of Figure 3 plots the cross-correlations between the market excess returns, $r_{m,t+1}^{ex}$, and leads ($j > 0$) and lags ($j < 0$) of tangible investment growth rates, ΔI_{t+1}. Consistent with the data, in our benchmark model the contemporaneous correlation between investment growth and excess returns is close to zero. This is the result of two offsetting effects. On the one hand, just like in the standard RBC model, a positive short-run productivity shock triggers a positive co-movement of the market return and investment growth. On the other hand, a positive long-run productivity shock boosts the market return but discourages current-period investment. In contrast, in model 1 long-run productivity shocks induce positive co-movements between investment growth and market returns and reinforce the effect originated from short-run shocks. As a result, model 1 produces a counterfactually large contemporaneous correlation of investment and market return.

In addition, similarly to the data, in our benchmark model the correlation reaches a peak for one-period-ahead investment and dies off at longer horizons. The correlation's surge at $j = 1$ is generated by two effects that reinforce each other. First, upon the realization of a positive short-run shock at time t, both intangible investment, J_t, and therefore S_{t+1}, rise. At time $t + 1$, because tangible investment and intangible capital

are complements, it is optimal to increase further tangible investment, I_{t+1}. Hence, a positive excess return at time t predicts a rise in investment at time $t+1$. Second, upon the realization of a positive long-run shock at time t, there is an immediate spike in the market excess return and a fall in physical investment followed by sluggish investment growth. Long-run shocks reinforce the fact that positive market excess returns at time t predict future positive investment growth starting from time $t+1$. Since these quantity dynamics die off over time, their effects on long-horizon correlations taper as well. In contrast, in model 1 the correlation between current excess returns and future investment growth is too high when $j=1$, and it quickly becomes negative at longer horizons.

In the right panel of Figure 3, we plot the cross-correlations between the return on the value-minus-growth portfolio, HML, and leads and lags of investment growth. Consistent with the data, the correlation between investment growth and HML is low for $j=0$ and increases gradually over longer horizons. Note that the returns on tangible and intangible capital move in the same direction after short-run shocks, but in opposite directions following long-run shocks; therefore, the HML return mainly reflects realizations of long-run productivity shocks in our model. As already noted, positive long-run shocks induce a small contemporaneous drop in physical investment followed by prolonged investment growth. Therefore, HML predicts future investment growth even though it has a negative contemporaneous correlation with investment.

In contrast, in model 1, the correlation between investment and HML return is too high for $j=0,1$ and quickly becomes negative at longer horizons. Overall, in our benchmark model, the correlations are consistently within the 95% confidence interval bands estimated from the data. We view the empirical evidence presented in this section as strongly supporting the economic mechanism emphasized by our model.

V Extensions

In this section we consider three extensions of our model and study their implications for a broader set of moments of macroeconomic quantities and asset prices. First, we introduce adjustment costs in the production of intangibles. Second, we add an endogenous labor supply. Finally, we further enrich the model and consider more general specifications of the ϕ_j process that governs the heterogeneity of firms' exposure to aggregate shocks. We show that our main results are preserved and often enhanced in these more general settings. For the sake of brevity, we focus our discussion only on moments that significantly change across the new extensions.

A Adjustment Costs

In Figure 4, the impulse responses of both tangible and intangible investment in the benchmark model contain a periodic component. As a result, in our benchmark model the volatility of intangible investment is about two times that of tangible investment, while this number is about one-half in the data (Table 9).

Here we introduce adjustment costs on the accumulation of intangible capital. This modification eliminates the periodic component in both tangible and intangible investments and makes their volatility consistent with the data. Specifically, we replace the law of motion of intangible capital in equation (8) by the following expression:

$$S_{t+1} \;\; = \;\; (1 - \delta_S)(S_t - G(I_t, S_t)) + H(J_t, K_t), \tag{29}$$

and parameterize H in the spirit of Jermann (1998):

$$H(J, K) = \left[\frac{a_1}{1 - 1/\xi} \left(\frac{J}{K} \right)^{1 - 1/\xi} + a_2 \right] K.$$

We calibrate the parameter ξ to match the volatility of intangible investment. Once ξ is chosen, the parameters $\{a_1, a_2\}$ are pinned down by the following two steady-state conditions: $H(\overline{J}, \overline{K}) = \overline{J}$ and $H_J(\overline{J}, \overline{K}) = 1$, where $\overline{J}$ and $\overline{K}$ denote the steady-state levels of intangible investment and tangible capital stock, respectively. In Appendix C.2 , we provide a microeconomic foundation for our specification of the adjustment cost function and prove that it arises as the result of a concave production function of new blueprints.

Given this modification of the model, the equilibrium conditions (14)–(16) are replaced by

$$
\begin{aligned}
q_{S,t} &= 1/H_{J,t} \\
p_{K,t} &= \alpha K_t^{\alpha-1} \left(A_t N_t \right)^{1-\alpha} + H_{K,t} q_{S,t} + (1 - \delta_K) q_{K,t} \\
r_{K,t+1} &= \frac{\alpha K_{t+1}^{\alpha-1} \left(A_{t+1} N_{t+1} \right)^{1-\alpha} + H_{K,t} q_{s,t} + (1 - \delta_K) q_{K,t+1}}{q_{K,t},} \\
r_{S,t+1} &= \frac{G_S \left(I_{t+1}, S_{t+1} \right)}{G_I \left(I_{t+1}, S_{t+1} \right)} + (1 - \delta_S) q_{S,t},
\end{aligned}
$$

where H_J and H_K denote the partial derivative of H with respect to J and K, respectively.

We highlight three main results. First, the impulse response functions of tangible and intangible invest-
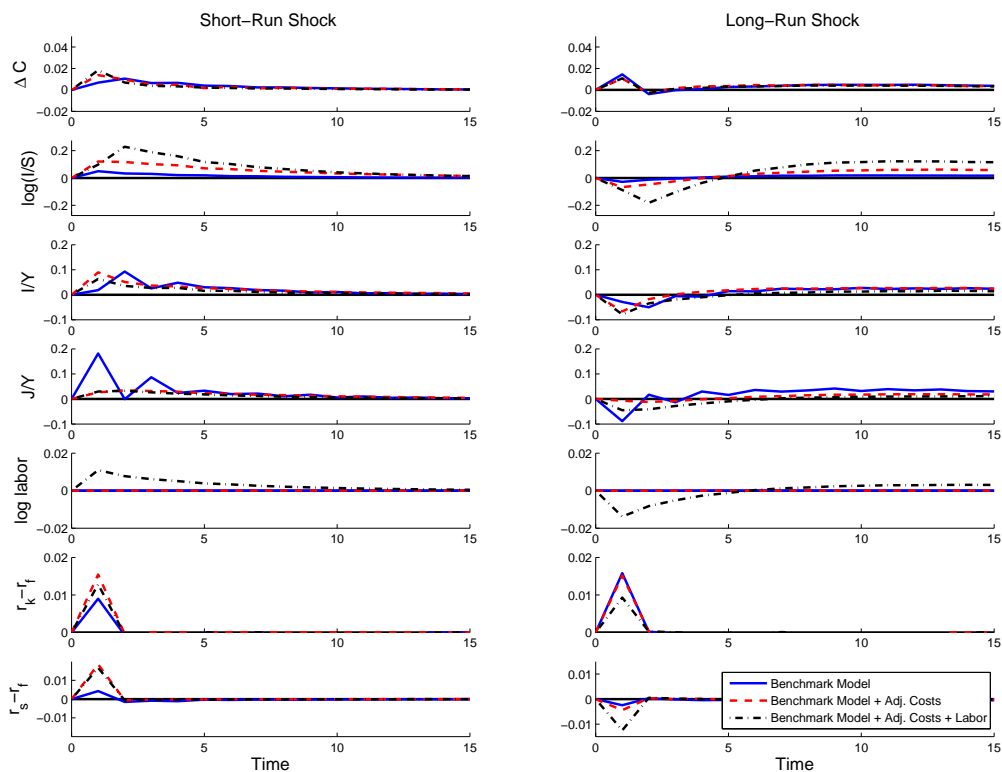
Figure 4 – Impulse Response Functions across Model Extensions

This figure shows annual log-deviations from the steady state. All the parameters are calibrated to the values reported in Table C.2 in Appendix C.

ment in the model with adjustment costs are smooth (Figure 4). Second, the incorporation of adjustment costs raises the volatility of physical investment and lowers that of intangible investment, consistent with the data (Table 9). Third, in the model with adjustment costs the behavior of consumption growth and returns on both tangible and intangible capital remains similar to that in the benchmark model. As a result, the implications of benchmark model for the equity premium, the value premium and the volatility of consumption growth are largely unaffected by this extension. Also, we observe that adding adjustment costs enhances the failure of the CAPM and produces a spread in the CAPM alphas consistent with the data (Table 9, last column).

We conclude this section with a brief explanation of the origin of the periodic component and the reason it disappears with adjustment costs. Because of the complementarity between intangible capital and physical investment, $G_{IS}(I_t, S_t) > 0$, the agent has a strong incentive to substantially increase physical investment

only after more intangible capital is built up. At time t, however, the stock of intangible capital, S_t, is one-period predetermined and cannot simultaneously adjust with tangible investment, I_t. For this reason, the agent finds it optimal to proceed in alternating steps.

For the sake of simplicity, let us focus on a positive short-run productivity shock at time $t = 1$ (Figure 4, left column). Upon the realization of the shock, physical investment is partially delayed and intangible investment, J_1, is immediately adjusted in order to reach a higher level of intangible stock, S_2. At time $t = 2$, physical investment is efficiently increased and intangible capital is partially depleted. At time $t = 3$, more intangible investment is needed to replenish the stock of blueprints. As a result, in the third period physical investment is dampened, and intangible investment surges again. This pattern continues until it converges back to the steady state.

Note that the above investment policy requires large adjustments in intangible investment, J, and produces an intangible investment growth volatility of 35% per year. Even in the presence of mild adjustment costs, such large changes in intangible capital become very costly. For this reason, with adjustment costs the increase in intangible investment becomes gradual and persistent, while physical investment immediately spikes upon the realization of the productivity shock. Because the incorporation of adjustment costs significantly improves our results, we keep this feature in the following two extensions as well.

B   Endogenous labor supply

Extension of the Model.   In this section, we allow for an endogenous labor supply and explore the ability of the model to account for the joint dynamics of aggregate consumption, investment, and hours worked. We report conventional business cycle statistics generated by the model in Table 9 and illustrate the response of macroeconomic quantities to productivity shocks in Figure 4.

To allow for endogenous labor supply, we adopt a Cobb-Douglas aggregator between consumption goods, C_t, and leisure, $1 - N_t$:

$$u_t = C_t^o (1 - N_t)^{1-o}.$$

We set the parameter o so that the average number of hours worked is equal to one-third of the total number

Table 9: Key Moments across Several Model Extensions

	$\sigma_{\Delta C}$	$\sigma_{\Delta I}/\sigma_{\Delta C}$	$\sigma_{\Delta J}/\sigma_{\Delta I}$	$\sigma_{\Delta n}$	$\rho_{\Delta C,\Delta n}$	$E[r_M^L - r^f]$	$E[r_K^L - r_S^L]$	$\alpha_K - \alpha_S$
Data	02.53	05.29	00.50	02.07	00.28	05.71	04.32	04.01
	(00.56)	(00.50)	(00.07)	(00.21)	(00.07)	(02.25)	(01.39)	(01.77)
Bench.	02.60	05.40	02.50	–	–	05.20	04.20	00.40
Ext. 1	02.53	06.40	00.40	–	–	04.55	04.16	04.91
Ext. 2	02.56	06.00	00.60	01.92	00.08	04.00	04.76	05.98
Ext. 3	02.59	06.23	00.50	01.84	00.28	05.41	05.68	07.70
Ext. 4	02.66	05.40	00.58	01.70	00.19	04.45	05.92	08.00

Notes - All figures are multiplied by 100, except contemporaneous correlations (denoted by ρ). Empirical moments are computed using US annual data in log units. Numbers in parentheses are GMM Newey-West adjusted standard errors. $E\left[r_K^L - r_S^L\right]$ and $E\left[r_M^L - r^f\right]$ measure the levered spread between tangible and intangible capital returns, and the market premium, respectively. The leverage coefficient is 3 (Garca-Feijo and Jorgensen (2010)). All the parameters are calibrated as in Table C.2 in Appendix C. The difference in the intercept of the CAPM regression for tangible and intangible returns is denoted by $\alpha_K - \alpha_S$. The entries for the models are obtained by repetitions of small-sample simulations. Extension 1 features adjustment costs on intangible investment. In Extension 2 we also add endogenous labor. In Extensions 3 and 4 we retain adjustment costs and endogenous labor and set ϕ_1 to 0 and 0.5, respectively, to study two-period transitions of productivity exposure.

of workable hours. The intratemporal first-order condition for labor is

$$\frac{1-o}{o} \cdot \frac{C_t}{1-N_t} = (1-\alpha)\frac{Y_t}{N_t},$$

and the stochastic discount factor becomes

$$\Lambda_{t+1} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-1} \left(\frac{u_{t+1}}{u_t}\right)^{1-\frac{1}{\psi}} \left[\frac{V_{t+1}}{E_t\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right]^{\frac{1}{\psi}-\gamma}.$$

All other equations that characterize the equilibrium remain unchanged.

Quantitative Results. First, we note that in Figure 4 the impulse response of consumption, investments, and returns are qualitatively similar to that obtained with an inelastic labor supply and adjustment costs on intangible capital. As a result, the model's implications for asset prices and macroeconomic quantities discussed so far remain largely unchanged (Table 9).

Second, in Figure 4, short-run shocks (contemporaneous productivity shocks) induce co-movement among consumption, investment, and hours worked, as in standard RBC models. Upon the realization of positive

long-run shocks (news about future productivity shocks), however, both investment and labor drop while consumption increases. The negative response of labor with respect to news is due to the income effect: good news about future productivity does not raise the current-period marginal product of labor but does increase the wealth of the agent. As a result, the agent works less, consumes more, and lowers investment. This feature of our model is consistent with the empirical evidence reported in Barsky and Sims (2010) and Kurmann and Otrok (2010) and enables us to produce a low contemporaneous correlation between consumption and labor growth. In this extension, $corr(\Delta C_t, \Delta n_t)$ is slightly lower than in the data, but it increases in the two model specifications to be discussed in the next subsection.

The reduction in the labor supply in response to news shocks lowers the marginal product of physical capital and enhances the decline in investment observed in the benchmark model. Consequently, intangible capital provides even more insurance against news shocks. This explains why the model with an elastic labor supply generates a slightly higher value premium, as reported in Table 9.

Overall, the inclusion of an endogenous labor supply preserves the success of previous versions of our model and generates plausible cyclical dynamics of hours worked, similar to the standard RBC model.

C More General Productivity Processes

Extension of the Model.   In this section, we consider an additional extension that allows for a more flexible specification of the heterogeneity in the productivity processes of production units. As we show in Section III and Appendix B, the exposure to aggregate productivity shocks is increasing in capital vintage age. Allowing for a gradual transition from low to high exposure requires more general specifications of the ϕ_j process and renders the aggregation results in equation (9) invalid. Intuitively, multiple transition periods introduce heterogeneity and history dependence of the productivity exposures and require us to keep track of the age distribution as a state variable.

To avoid computational complexity, we restrict our attention to the following simple form of the ϕ_j function:

$$\phi_j = \begin{cases} 0 & j = 0 \\ \phi_1 & j = 1 \\ 1 & j = 2, \dots \end{cases},$$

which allows the transition to happen in two periods. Previous versions of the model correspond to the special case with $\phi_1 = 1$.

In this case, the social planner's problem can be made recursive by including last-period physical investment as an additional state variable. To this aim, we use X_t to denote the total measure of production units constructed at time $t - 1$ and continue to use K_t to denote the productivity-adjusted stock of production units at time t. The laws of motion of K_t and X_t can be written as follows:

$$
\begin{align}
K_{t+1} &= (1 - \delta_K)K_t + X_t[\pi_{t+1} - 1] + \varpi_{t+1}G(I_t, S_t), \tag{30} \\
X_{t+1} &= (1 - \delta_K)\varpi_{t+1}G(I_t, S_t), \\
\pi_{t+1} &= e^{-\frac{1-\alpha}{\alpha}(1-\phi_1)(x_t + \sigma_a \varepsilon_{a,t+1})} \quad \forall t.
\end{align}
$$

The social planner's problem is the same as before except that we replace equation (9) with equation (30), and the value function, $V(K_t, X_t, S_t, x_t, A_t)$, now contains the additional state variable X_t.

Quantitative Results. We consider two calibrations of the ϕ_j function. In the first, we set $\phi_1 = 0$ so that new production units have zero exposure to aggregate productivity shocks for two periods. In the second calibration, we set $\phi_1 = 0.5$ so that the correlation with aggregate productivity shocks of new production units is 0 in its first period, 0.5 in its second period, and 1 from the third period on. We report our results in the last two rows of Table 9. Multiple transition periods enhance the positive exposure of the return on tangible capital and the negative exposure of return on intangible capital to long-run productivity shocks. Consequently, these extensions allow us to reduce the volatility of long-run productivity shocks relative to previous calibrations, while still maintaining high equity and value premiums. As a result, the negative correlation between consumption and hours worked induced by long-run shocks is dampened, and our model produces a strong co-movement of consumption and labor, closely matching this moment in the data. All other quantitative implications of the model remain largely unchanged.

VI   Conclusion

In this study, we present a general equilibrium asset pricing model with long-run productivity shocks as in Croce (2008) and intangible capital modeled as storable investment options as in Ai (2009). We document that in the data, new investment is less exposed to aggregate productivity shocks than is capital of older vintages. We incorporate this feature in our model and show that the lower exposure of new investment is

quantitatively important in accounting for (1) the high equity premium, (2) the high volatility of the stock market return, and (3) the large spread in both expected returns and cash-flow duration across book-to-market-sorted portfolios in the data.

Several remarks are in order. First, as in Ai (2009), we have allowed idiosyncratic shocks to the quality of investment options. In our setting, we have assumed that these shocks are *i.i.d.* While unrealistic, this assumption simplifies our aggregation results, making our model very tractable and enabling us to avoid the need to keep track of the cross-sectional distribution of option quality. Allowing for more general processes of the quality of the options is a fruitful extension that we leave for future research.

Second, considering a more general setting with heterogeneous firms will allow us to implement portfolio sorting exercises in the context of our current model, to study firms' transition among value and growth portfolios, and to confront the model with a wider set of empirical evidence at the portfolio level, as done by Ai and Kiku (2009). Based on their insights, we are optimistic that the basic intuition in this paper will remain valid even with heterogeneous firms.

Finally, we believe that our model provides a valuable general equilibrium framework for the measurement of intangible capital by exploring the information from both the quantity and pricing sides of the economy. Specifically, a structural estimation of our DSGE model employing both time-series data on macroeconomic aggregates and cross-section data on equity returns may shed new light on the accumulation of intangibles in the US.

References

Ai, H. (2009). Intangible capital and the value premium. Working paper.

Ai, H. and D. Kiku (2009, August). Growth to value: Option exercise and the cross-section of equity returns. Working paper.

Altman, E. I. (2000). Predicting fiancial distress of companies: Revisiting the z-score and zeta models. Working paper.

Bansal, R., R. Dittmar, and C. Lundblad (2005, August). Consumption, dividends, and the cross section of equity returns. *The Journal of Finance 60*(4), 1639–1672.

Bansal, R., D. Kiku, and A. Yaron (2007). Risks for the long run: Estimation and inference. *Working Paper.*.

Bansal, R. and A. Yaron (2004, August). Risk for the long run: A potential resolution of asset pricing puzzles. *The Journal of Finance 59*(4), 1481–1509.

Barsky, B. and E. Sims (2010). News shocks and business cycles. *Working paper*.

Berk, J., R. C. Green, and V. Naik (1999). Optimal investment, growth options, and security returns. *The Journal of Finance 54*, 1553–1608.

Blundell, R. W. and S. R. Bond (2000). Gmm estimation with persistent panel data: An application to production functions. *Econometric Reviews*.

Borovicka, J. and L. Hansen (2011). Examining macroeconomic models through the lens of asset pricing. *Working Paper*.

Borovicka, J., L. Hansen, M. Hendricks, and J. Scheinkman (2011). Risk-price dynamics. *Journal of Financial Econometrics*.

Campbell, J. (1999). *Asset Prices, Consumption, and the Business Cycle*, Volume 1 of *Handbook of Macroeconomics*. North-Holland: Amsterdam.

Campbell, J. (2000, August). Asset pricing at the millennium. *Journal of Finance 55*, 1515–1567.

Carlson, M., A. Fisher, and R. Giammarino (2004, December). Corporate investment and asset price dynamics: Implications for the cross-section of returns. *The Journal of Finance 59*(6), 2577–2603.

Cooper, I. (2006, February). Asset pricing implications of nonconvex adjustment costs and irreversibility of investment. *The Journal of Finance 61*(1), 139–170.

Corrado, C., C. Hulten, and D. Sichel (2006). Intangible capital and economic growth. Working paper.

Croce, M. M. (2008). Long-run productivity risk: A new hope for production-based asset pricing. Working paper.

Da, Z. (2006). Cash flow, consumption risk, and the cross-section of stock returns. *forthcoming in The Journal of Finance*.

Da, Z., R.-J. Guo, and R. Jagannathan (2012). Capm for estimating the cost of equity capital: Interpreting the empirical evidence. *Journal of Financial Economics*.

Dechow, P., R. Sloan, and M. Soliman (2004). Implied equity duration: A new measure of equity risk. *Review of Accounting Studies*, 197–228.

Eisfeldt, A. and D. Papanikolaou (2009). Organization capital and the cross-section of expected returns. Working paper.

Epstein, L. and S. E. Zin (1989, July). Substitution, risk aversion, and the temporal behavior of consumption and asset returns: A theoretical framework. *Econometrica 57*(4), 937–969.

Fama, E. and K. R. French (1992, June). The cross section of expected stock returns. *The Journal of Finance XLVII*(2), 427–465.

Fama, E. and K. R. French (1995, March). Size and book-to-market factors in earnings and returns. *The Journal of Finance 50*(1), 135–155.

Gala, V. (2005). Investment and returns.

Garca-Feijo, L. and R. D. Jorgensen (2010). Can operating leverage be the cause of the value premium? *Financial Management*.

Gomes, J., L. Kogan, and L. Zhang (2003, August). Euqilibrium cross section of returns. *Journal of Political Economy 111*(4).

Gourio, F. and L. Rudanko (2010). Customer capital. *Working paper*.

Greene, W. H. (2002). Econometric analysis. *Prentice Hall*.

Hall, R. (2001, December). The stock market and capital accumulation. *American Economic Review 91*(5), 1185–1202.

Hand, J. and B. Lev (2003). *Intangible Assets*. Oxford Management Readers.

Hansen, L., J. Heaton, and N. Li (2005). *Intangible Risk?* Measuring Capital in the New Economy (NBER Books).

Hansen, L. P., J. C. Heaton, and N. Li (2008). Consumption strikes back? measuring long-run risk. *Journal of Political Economy 116*(2).

Heckman, J. (1979). Sample selection bias as a specification error. *Econometrica 47*(1), 153–162.

Jermann, U. J. (1998, April). Asset pricing in production economies. *Journal of Monetary Economics 41*(2), 257–275.

Jovanovic, B. (2008, May). Investment options and the business cycle. *Journal of Economic Theory*.

Jovanovic, B. and P. Rousseau (2001). Why wait? a century of life before ipo. *American Economic Review, P&P*.

Kaltenbrunner, G. and L. Lochstoer (2010). Long-run risk through consumption smoothing. *Forthcoming on the Review of Financial Studies.*.

Kiku, D. (2006). Long run risks and the value premium puzzle. Working paper, Duke University.

Kogan, L. and D. Papanikolaou (2009). Growth opportunities, technology shocks and asset prices. Working paper.

Kogan, L. and D. Papanikolaou (2010). Growth opportunities and technology shocks. *American Economic Review, P&P*.

Kogan, L. and D. Papanikolaou (2011). Economic activity of firms and asset prices.

Kreps, D. M. and E. L. Porteus (1978, January). Temporal resolution of uncertainty and dynamic choice theory. *Econometrica 46*(1), 185–200.

Kurmann, A. and C. Otrok (2010). News shocks and the slope of the term structure of interest rates. *Working paper*.

Kydland, F. and E. Prescott (1982, November). Time to build and aggregtae fluctuations. *Econometrica 50*(6), 1345–1370.

Li, E. and L. Liu (2010). Intangible assets and cross-sectional stock returns: Evidence from structural estimation. Working paper.

Li, N. (2009). Intangible capital and stock prices.

Lin, X. (2009). Endogenous technological progress and the cross section of stock returns. *Working paper*.

McGrattan, E. R. and E. C. Prescott (2009a). Technology capital and the u.s. current account. *American Economic Review, forthcoming*.

McGrattan, E. R. and E. C. Prescott (2009b). Unmeasured investment and the puzzling u.s. boom in the 1990s. *American Economic Journal: Macroeconomics, forthcoming*.

Mehra, R. and E. Prescott (1985, March). The equity premium: A puzzle. *Journal of Monetary Economics 15*(2), 145–161.

Pastor, L. and P. Veronesi (2009). Technological revolutions and stock prices. *American Economic Review 99(4)*.

Ritter, J. and T. Loughran (2004). Why has ipo underpricing changed over time? *Financial Management 33*(3).

Santos, T. and P. Veronesi (2010). Habit formation, the cross section of stock returns and the cash flow risk puzzle. *Journal of Financial Economics*.

Swanson, E. (2012). Risk aversion and the labor margin in dynamic equilibium models. *American Economic Review*. Forthcoming.

Tallarini, T. D. (2000, June). Risk sensitive real business cycles'. *Journal of Monetary Economics 45*(3), 507–532.

Weil, P. (1989, January). The equity premium puzzle and the riskfree rate puzzle. *Journal of Monetary Economics 24*(3), 401–422.

Zhang, L. (2005, February). The value premium. *The Journal of Finance 60*(1), 67–103.

Appendix A: Aggregation of Production Units

Lemma 1 *Suppose there are m types of firms. For $i = 1, 2, 3, \cdots m$, the productivity of the type i firm is denoted by $A(i)$, and the total measure of the type i firm is denoted by $K(i)$. The production technology of the type i firm is given by*

$$y(i) = [A(i) n(i)]^{1-\alpha},$$

where $n(i)$ denotes the labor hired at firm i. The total labor supply in the economy is N. Then total output is given by

$$Y = \left[\sum_{i=1}^{m} K(i) \left[\frac{A(i)}{A(1)} \right]^{\frac{1-\alpha}{\alpha}} \right]^{\alpha} [A(1) N]^{1-\alpha}.$$

Proof. *Without loss of generality, we assume that firms of the same type employ the same amount of labor. In this case, the total output in the economy is given by*

$$Y = \max \sum_{i=1}^{m} K(i) A(i)^{1-\alpha} n(i)^{1-\alpha} \tag{A.1}$$

$$\text{subject to } \sum_{i=1}^{m} K(i) n(i) = N$$

The first-order condition of the above optimization problem implies that for all i,

$$\frac{n(i)}{n(1)} = \left(\frac{A(i)}{A(1)} \right)^{\frac{1-\alpha}{\alpha}}$$

Using the resource constraint, we determine the labor employed in firm 1:

$$\sum_{i=1}^{m} K(i) \left(\frac{A(i)}{A(1)} \right)^{\frac{1-\alpha}{\alpha}} n(1) = N$$

This implies that

$$n(1) = \left[\sum_{i=1}^{m} K(i) \left[\frac{A(i)}{A(1)} \right]^{\frac{1-\alpha}{\alpha}} \right]^{-1} N \tag{A.2}$$

Therefore, the total production is given by

$$Y = \sum_{i=1}^{m} K(i) A(i)^{1-\alpha} \left[\left(\frac{A(i)}{A(1)} \right)^{\frac{1-\alpha}{\alpha}} n(1) \right]^{1-\alpha}$$

$$= \left[A(1)^{-\frac{1-\alpha}{\alpha}} n(1) \right]^{1-\alpha} \sum_{i=1}^{m} K(i) A(i)^{\frac{1-\alpha}{\alpha}}$$

$$= \left[\sum_{i=1}^{m} K(i) A(i)^{\frac{1-\alpha}{\alpha}} \right]^{\alpha} N^{1-\alpha}$$

$$= A(1) \left[\sum_{i=1}^{m} K(i) \left(\frac{A(i)}{A(1)} \right)^{\frac{1}{\alpha}} \right]^{\alpha} N^{1-\alpha}$$

Plugging in the expression for $n(1)$ in equation (A.2), we have

$$Y = \left[A(1)^{-\frac{1-\alpha}{\alpha}} \left[\sum_{i=1}^{m} K(i) \left[\frac{A(i)}{A(1)} \right]^{\frac{1-\alpha}{\alpha}} \right]^{-1} N \right]^{1-\alpha} \sum_{i=1}^{m} K(i) A(i)^{\frac{1-\alpha}{\alpha}}$$

$$= \left[\sum_{i=1}^{m} K(i) A(i)^{\frac{1-\alpha}{\alpha}} \right]^{\alpha} N^{1-\alpha}$$

$$= \left[\sum_{i=1}^{m} K(i) \left[\frac{A(i)}{A(1)} \right]^{\frac{1-\alpha}{\alpha}} \right]^{\alpha} A(1)^{1-\alpha} N^{1-\alpha}$$

as needed. ∎

At time t, there are $t+1$ types of operating production units in the economy, namely, production units of generation $-1, 0, 1, \cdots, t-1$. The measures of these production units are $(1-\delta_K)^t K_0$, $(1-\delta_K)^{t-1} M_0$, $(1-\delta)^{t-2} M_1, \cdots, M_{t-1}$. Using the above lemma, at date t, the total production in the economy is given by

$$Y_t = A_t \left[(1-\delta_K)^t K_0 + \sum_{j=0}^{t-1} (1-\delta_K)^{t-j-1} E_j \left(\frac{A_t^j}{A_t} \right)^{\frac{1-1}{\alpha}} \right]^{\alpha} N_t^{1-\alpha}.$$

Clearly, if we define the $\{K_t\}_{t=0}^{\infty}$ according to (9), the aggregate production function can be summarized as in (5).

Appendix B: Estimation Details and Aggregate Data Sources

B.1 Robustness Analysis for Firm-level Regressions

Endogeneity and the Dynamic Error Component Model. We follow Blundell and Bond (2000) and write the firm-level production function as follows:

$$\ln y_{i,t} = z_i + w_t + \alpha_1 \ln k_{i,t} + \alpha_2 \ln n_{i,t} + v_{i,t} + u_{i,t} \tag{B.1}$$
$$v_{i,t} = \rho v_{i,t-1} + e_{i,t},$$

where z_i, w_t and $v_{i,t}$ indicate a firm fixed effect, a time-specific intercept, and a possibly autoregressive productivity shock, respectively. The residuals from the regression are denoted by $u_{i,t}$ and $e_{i,t}$ and are assumed to be white noise processes. The model has the following dynamic representation:

$$\Delta \ln y_{i,j,t} = \rho \Delta \ln y_{i,j,t-1} + \alpha_{1,j} \Delta \ln k_{i,j,t} - \rho \alpha_{1,j} \Delta \ln k_{i,j,t-1} + \alpha_2 \Delta \ln n_{i,j,t} - \rho \alpha_2 \Delta \ln n_{i,j,t-1} \tag{B.2}$$
$$+ (\Delta w_t - \rho \Delta w_{t-1}) + \Delta \kappa_{i,t},$$

where $\kappa_{i,t} = e_{i,t} + u_{i,t} - \rho u_{i,t-1}$. Let $x_{i,j,t} = \{\ln (k_{i,j,t}), \ln (n_{i,j,t}), \ln (y_{i,j,t})\}$. Assuming that $E[x_{i,j,t-l} e_{i,t}] = E[x_{i,j,t-l} u_{i,t}] = 0$ for $l > 0$ yields the following moment conditions:

$$E[x_{i,i,t-l} \Delta \kappa_{i,t}] = 0 \text{ for } l \geq 3$$
$$E[\Delta x_{i,i,t-l} \Delta \kappa_{i,t}] = 0 \text{ for } l \geq 3.$$

that are used to conduct a consistent GMM estimation of (B.2). Given the estimates $\widehat{\alpha}_{1,j}$ and $\widehat{\alpha}_{2,j}$, log productivity of firm i is computed as:

$$\ln \widehat{a}_{i,j,t} = y_{i,j,t} - \widehat{\alpha}_{1,j} k_{i,j,t} - \widehat{\alpha}_{2,j} n_{i,j,t}. \tag{B.3}$$

We apply this method for regressions (5)–(6) and (9)–(10) of Table B.1. In all specifications, the correlation between firm and aggregate productivity is increasing in capital age, consistent with our main results reported in Table 2.

Endogeneity and fixed effects. An alternative way to estimate the production function avoiding endogeneity issues is to work with the following regression:

$$\ln y_{i,j,t} = v_j + z_{i,j} + w_{j,t} + \alpha_{1,j} \ln k_{i,j,t} + \alpha_{2,j} \ln n_{i,j,t} + u_{i,j,t}. \tag{B.4}$$

The parameters v_j, $z_{i,j}$, and w_{jt} indicate an industry dummy, a firm fixed effect, and an industry-specific time dummy, respectively. The residual from the regression is denoted by $u_{i,j,t}$. Given our point estimate of $\widehat{\alpha}_{1,j}$ and $\widehat{\alpha}_{2,j}$, we can use equation (B.3) to estimate $\Delta a_{i,j,t}$. Given this estimation of firms' productivity, we proceed as before in estimating equation (19). The results are summarized in regression (1)–(4) and (7)–(8)

Table B.1: Exposure to Aggregate Risk by Firm Age

Regr.	Age	Sample	E.M.	ξ_3	ξ_4	Obs.	Firms
(1)	F	All	FE	0.011***	-0.003	70899	7333
(2)	F	$\Delta a_t > 0$	FE	0.013***	-0.004	59385	7224
(3)	K-5	All	FE	0.426**	-0.120***	82791	8012
(4)	K-5	$\Delta a_t > 0$	FE	0.880**	-0.133***	63717	7828
(5)	K-5	All	ECM	0.494**	-0.146***	82787	8010
(6)	K-5	$\Delta a_t > 0$	ECM	0.862**	-0.159***	63713	7826
(7)	K-15	All	FE	0.383***	-0.014***	32127	2928
(8)	K-15	$\Delta a_t > 0$	FE	0.511***	-0.016***	25955	2876
(9)	K-15	All	ECM	0.425***	-0.019***	32130	2929
(10)	K-15	$\Delta a_t > 0$	ECM	0.510***	-0.022***	25958	2877

Notes - This table reports firms' risk exposure by age. All estimates are based on the following second-stage regression: $\Delta \ln a_{i,j,t} = \xi_{0i} + \xi_1 \Delta \ln A_t + \xi_2 AGE_{i,j,t} + \xi_3 AGE_{i,j,t} \cdot \Delta \ln A_t + \xi_4 B/M_{i,j,t} + \varepsilon_{ijt}$. In col. "Age", F, K-5 and K-15 denote firm age, capital age with T=5, and capital age with T=15, respectively. All regressions denoted by an even number are computed on a sub-sample including only years of positive productivity growth to control for firms exit. "E.M." stands for Estimation Method used in the first stage to estimate $\Delta a_{i,j,t}$. We use FE to denote our Fix Effect procedure described in (B.4) and ECM for the dynamic Error Component Model of Blundell and Bond (2000) described in (B.2). We use $*$, $**$, and $***$ to indicate p-values smaller than 0.105, 0.05, and 0.01, respectively. Standard errors available upon request.

of Table B.1 and are consistent with those reported in Table 2.

Sample Selection Bias. If exits caused by exposure to negative aggregate productivity shocks are correlated with firm age, they can induce an upward bias in our estimate of ξ_3, the coefficient that measures variation in productivity exposure due to age. We correct for sample selection bias in Regressions (2) and (3) of Tables 2 and 3. Our result that firms' exposure to aggregate productivity shocks is increasing in age is robust even after we control for potential sample selection bias.

We implement the Heckman (1979) two-stage procedure in regression (2). First, we project an indicator variable of firms' exit on a vector of observables, including the Altman (2000) Score, size (measured by total book value of assets in real terms), size squared, R&D expenditure-sales ratio, earnings over sales, capital-labor ratio, and aggregate productivity growth. Second, we compute the implied Inverse Mills Ratio (IMR) (see Greene (2002)) and include it as an additional independent variable in regression (19). In regression (3), we include observations only in years with positive aggregate productivity shocks. Overall, our point estimate for ξ_3 is positive and significant across all specifications.

B.2 Aggregate Data Sources

Consumption (C_t). Per capita consumption data are from the National Income and Product Accounts (NIPA) annual data reported by the Bureau of Economic Analysis (BEA). The data are constructed as the sum of consumption expenditures on nondurable goods and services (Table 1.1.5, lines 5 and 6) deflated by corresponding price deflators (Table 1.1.9, lines 5 and 6).

Physical Investment (I_t). Physical investment data are also from the NIPA tables. We measure physical investment by fixed investment (Table 1.1.5, line 8) minus information processing equipment and software (Table 5.5.5, line 3) deflated by its price deflator (Table 1.1.9, line 8). Information processing equipment and software is interpreted as investment in intangible capital and is therefore subtracted from fixed investment.

Measured Output ($Y_{M,t}$). It is the sum of total consumption and physical investment, that is, $C_t + I_t$. We exclude government expenditure and net export because not explicitly modelled in our economy. Notice also that the NIPA tables do not account for J_t over the sample 1929–2003.

Intangible Investment (J_t). We follow the procedure in Corrado et al. (2006) to construct a measure of intangible investment from 1953 to 2003. We include expenses in brand equity, firm-specific resources, R&D and computerized information. Prior to 1953 data are not available. Data and notes on our sources are available upon request.

Labor (N_t). It is measured as the total number of full-time and part-time employees as reported in the NIPA table 6.4. Data are annual from 1929 to 2003. In table 9, we divide N by total population.

Physical Capital (K_t). We follow Hall (2001) and construct capital according to the following recursion: $K_t = (1 - \delta_K)K_{t-1} + I_t$, where $\delta_K = .11$ as in our calibration. This recursion is initialized in 1929 by imposing $K_{1929} = I_{1929}/\delta_K$ as in Hall (2001).

Asset returns. Both the market returns and the *HML* factor are from the Fama-French dataset available online on K. French's webpage:
`http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/ftp/F-F_Research_Data_Factors.zip`.

Risk-free rate. The nominal risk-free rate is measured by the annual 3-month T-Bill return. The real risk-free rate is computed by subtracting realized inflation from the nominal risk-free rate.

Appendix C: Calculation of the Macaulay Duration and Details of Model Extensions

C.1 Duration.

In this section, we derive a recursive relation that can be used to calculate the Macaulay duration of growth options and assets in place defined in (25). We first prove the following lemma.

Lemma 2 *Let MD_t and P_t be the time-t Macaulay duration and present value of the cash flow process $\{CF_t\}_{t=1}^{\infty}$, respectively; then*

$$MD_t \cdot P_t = E_t\left[\Lambda_{t+1} \cdot CF_{t+1}\right] + E_t\left[\Lambda_{t+1}(1 + MD_{t+1})P_{t+1}\right]. \tag{C.1}$$

Furthermore, if $CF_t = CF_{1,t} + CF_{2,t}$ for all t, then

$$
\begin{aligned}
MD_t \cdot P_t = E_t\left[\Lambda_{t+1}\left(CF_{1,t+1} + CF_{2,t+1}\right)\right] \quad &+ \quad E_t\left[\Lambda_{t+1}(1 + MD_{1,t+1})P_{1,t+1}\right]\dots \\
&+ \quad E_t\left[\Lambda_{t+1}(1 + MD_{2,t+1})P_{2,t+1}\right],
\end{aligned} \tag{C.2}
$$

where $P_{i,t}$ and $MD_{i,t}$ denote the price and Macaulay duration of cash flow $CF_{i,t}$ for $i = 1, 2$.
Proof. *By the definition of Macaulay duration,*

$$
\begin{aligned}
MD_t \cdot P_t \;&=\; E_t\left[\sum_{j=1}^{\infty} j\Lambda_{t,t+j}CF_{t+j}\right] \\
&=\; E_t\left[\Lambda_{t,t+1}CF_{t+1}\right] + E_t\left[E_{t+1}\left[\sum_{j=2}^{\infty} j\Lambda_{t,t+j}CF_{t+j}\right]\right] \\
&=\; E_t\left[\Lambda_{t,t+1}CF_{t+1}\right] \\
&\quad + E_t\left[\Lambda_{t,t+1}\left(E_{t+1}\left[\sum_{j=2}^{\infty}(j-1)\Lambda_{t+1,t+j}CF_{t+j}\right] + E_{t+1}\left[\sum_{j=2}^{\infty}\Lambda_{t+1,t+j}CF_{t+j}\right]\right)\right] \\
&=\; E_t\left[\Lambda_{t,t+1}\left(CF_{t+1} + MD_{t+1}P_{t+1} + P_{t+1}\right)\right],
\end{aligned}
$$

as needed.
Equation (C.1) can then be proved by applying the definition of present value and duration separately for cash flow $\{CF_{1,t}\}_{t=1}^{\infty}$ and $\{CF_{2,t}\}_{t=1}^{\infty}$. ∎

Let $MD_{S,t}$ denote the Macaulay duration of a growth option created at the end of period t. Let $MD_{K,t}$ refer to the Macaulay duration of a generation-0 production unit survived until the end of period t. We show

that $MD_{S,t}$ and $MD_{K,t}$ satisfy the following recursive relation:

$$MD_{S,t} \cdot q_{S,t} = -E_t\left[\Lambda_{t+1}\frac{I_{t+1}}{S_{t+1}}\right] \quad + \quad E_t\left[\Lambda_{t+1}\frac{G\left(I_{t+1},S_{t+1}\right)}{S_{t+1}}(1+MD_{K,t+1})q_{K,t+1}\varpi_{t+1}\right]\ldots \tag{C.3}$$

$$+ \quad E_t\left[\Lambda_{t+1}\left(1-\frac{G\left(I_{t+1},S_{t+1}\right)}{S_{t+1}}\right)(1+MD_{S,t+1})q_{S,t+1}(1-\delta_S)\right],$$

$$MD_{K,t} \cdot q_{K,t} = E_t\left[\Lambda_{t+1}\left(\alpha y_{t+1} - \frac{J_{t+1}}{K_{t+1}}\right)\right] \quad + \quad E_t\left[\Lambda_{t+1}(1-\delta_K)(1+MD_{K,t+1})q_{K,t+1}\right]\ldots \tag{C.4}$$

$$+ \quad E_t\left[\Lambda_{t+1}\frac{J_{t+1}}{K_{t+1}}(1+MD_{S,t+1})q_{S,t+1}\right].$$

Consider a growth option at the end of period t. In period $t+1$ after the quality of the option, θ, is revealed, there are two possibilities. If $\theta \geq \theta_{t+1}^* = G_I\left(I_t, S_t\right)$, then the option is exercised. In this case, the cash flow in period $t+1$ is $-\frac{1}{\theta}$, and the option becomes a generation $t+1$ production unit, which generates cash flow equivalent to ϖ_{t+1} generation-0 production units. In the case $\theta < G_I\left(I_t, S_t\right)$, the option is not exercised and survives to the next period with probability $1-\delta_S$, in which case it generate the cash flow of a period $t+1$ growth option. Note that the above argument provides a cash flow decomposition of a growth option at period t. By lemma 2, we have

$$MD_{S,t} \cdot q_{S,t} = E_t\left[\Lambda_{t,t+1} \cdot \int_{\theta \geq G_I(I_t,S_t)}\left[-\frac{1}{\theta} + (1+MD_{K,t+1})\varpi_{t+1}q_{K,t+1}\right]f\left(\theta\right)d\theta\right]$$

$$+E_t\left[\Lambda_{t,t+1} \cdot \int_{\theta < G_I(I_t,S_t)}\left[(1-\delta_S)\left(1+MD_{S,t+1}\right)q_{S,t+1}\right]f\left(\theta\right)d\theta\right].$$

Using the results in Ai (2009), the integrals can be written as functions of the aggregate quantities:

$$MD_{S,t} \cdot q_{S,t} = E_t\left[\Lambda_{t,t+1} \cdot \left[-\frac{I_{t+1}}{S_{t+1}} + \frac{G\left(I_{t+1},S_{t+1}\right)}{S_{t+1}}\left(1+MD_{K,t+1}\right)\varpi_{t+1}q_{K,t+1}\right]\right]$$

$$+E_t\left[\Lambda_{t,t+1} \cdot \frac{G\left(I_{t+1},S_{t+1}\right)}{S_{t+1}}\left[(1-\delta_S)\left(1+MD_{S,t+1}\right)q_{S,t+1}\right]\right],$$

as needed.

We can decompose the cash flow of a production unit as well. The total output of a period-t production unit is used for three purposes: consumption; tangible investment that produces new-generation production units; and intangible investment that creates new blueprints, which are associated with three difference sources of future cash flows. Equation (C.4) can then be established by applying lemma 2 to the above cash flow decomposition. By solving the system of recursive equations (C.3)–(C.4) we obtain the pair of Macaulay durations, $(MD_{K,t}, MD_{S,t})$.

C.2 Microeconomic foundation of the Adjustment Cost Function H

Here we show that the law of motion of intangible capital in equation (29) arises as the result of a concave production function of new blueprints. Suppose consumption goods, new blueprints, and new investment goods are produced in production units. Let c, i and j denote the amount of general output used to produce consumption goods, investment goods, and blueprints, respectively. Normalize the price of consumption goods to 1, and denote q_S and q_I the price of blueprints and investment goods, respectively. In this case, the profit maximization problem of a typical production unit can be written as:

$$\max_{c,i,j,n} \left[c + q_I i + q_S h(j) - wn \right]$$

$$c + i + j = (An)^{1-\alpha}.$$

In equilibrium we must have $q_I = 1$. In addition, optimality requires $q_S = 1/h'(j)$, which implies that all production units produce the same amount of blueprints. We continue to use K to denote the total measure of production units. The total amount of resources used to produce blueprints is therefore $J = j \cdot K$, and the total amount of blueprints produced is $K \cdot h(J/K)$ in this economy. After denoting $H(J, K) = h(J/K)K$, the law of motion of intangible capital can be written as in equation (29).

The function H, which resembles adjustment costs in neoclassical models, is homogenous of degree one and concave in (J, K). Accordingly, we specify H in the spirit of Jermann (1998) as follows:

$$H(J, K) = \left[\frac{a_1}{1 - 1/\xi} \left(\frac{J}{K} \right)^{1-1/\xi} + a_2 \right] K,$$

where $1/\xi$ determines the concavity of H and the parameters $\{a_1, a_2\}$ satisfy the following two steady state conditions: $H(\overline{J}, \overline{K}) = \overline{J}$ and $H_J(\overline{J}, \overline{K}) = 1$.

C.3 Linking the Aggregator G to Microeconomic Evidence

In Section IV.A of the paper, we calibrate the CES aggregator, $G(I, S)$, to match macroeconomic moments. Here we describe a procedure that links the functional form of $G(I, S)$ to microeconomic evidence on the market-to-book ratio of new Initial Public Offering (IPO) firms.

The theoretical link between G and the market-to-book ratio of exercised options. Using the results in Ai (2009), the CES aggregator, $G(I, S)$ implies that $\theta_{i,t}$ is an i.i.d. draw from the following distribution:

$$P(\theta_{i,t} \leq \theta) = \int_0^\theta \frac{\nu^{-\xi} x^{\xi-1}}{[\nu^{-\xi} x^{\xi-1} - 1]^{1 + \frac{1}{1-1/\xi}}} dx, \quad \theta \in [0, +\infty). \tag{C.5}$$

In our model, the time-t market value of a newly created production unit is $\varpi_t p_{K,t}$. Its book value is the value of investment goods used to implement the blueprint in the last period, $\frac{1}{\theta_{i,t-1}}$. Therefore the market-to-
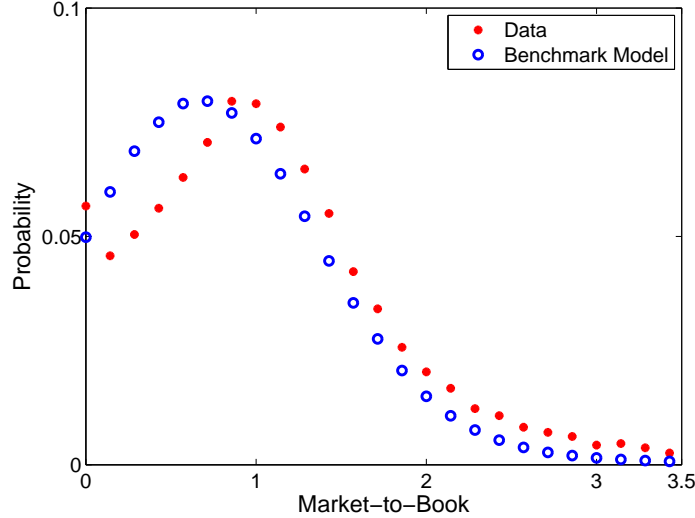
Figure C.4 – Book-to-Market Distribution for New Production Units.
Notes - The circles refer to our Benchmark Model, while the stars refer to IPO.s data from the CDS data set. Our annual sample starts in 1970 and ends in 2009 and includes 22,116 observations.

book ratio of a new production unit at time t is $\theta_{i,t-1}\varpi_t p_{K,t}$. Note that not all blueprints are implemented. By Proposition 1, a blueprint i is implemented in period $t-1$ if and only if $\theta_{i,t-1} \geq \theta_{t-1}^* = G_I\left(I_{t-1}, S_{t-1}\right)$.

An empirical proxy for the market-to-book ratio of newly exercised options. The above argument links the truncated density f to the distribution of market-to-book ratios of newly exercised options across firms at a given time. The market-to-book ratios of the firms in our COMPUSTAT-CRSP data set, in contrast, reflect the market-to-book ratio of options exercised by the same firm at different points in times. For this reason, we consider the market-to-book ratio of new IPO firms a better proxy for that of newly exercised options. We think of implementation of blueprints as initial public offering, and we compare the cross-section distribution of the market-to-book ratio of newly exercised options in our simulation with that of the new IPO firms in the SDC Platinum data set.

Details of the Simulation Procedure. In our simulation, we allow the productivity of production units to have an idiosyncratic component ε_t^i:

$$A_{i,t}^t = A_t^t \cdot \varepsilon_t^i.$$

We set $E\left[\varepsilon_t^i\right] = 1$ and choose $Var\left[\varepsilon_t^i\right]$ to match the cross-sectional dispersion of productivity in our COM-PUSTAT data. In this case, all aggregation results in our model remain unchanged. The only difference is that the market-to-book ratio of firms i becomes $\varepsilon_t^i\theta_{i,t-1}\varpi_t p_{K,t}$.

We simulate the time series of macroeconomic quantities from our model. Note that S_t measures the

Table C.2 Calibration for Model Extensions

Extension		1	2	3	4
Preference parameters					
Discount factor	β	0.97	0.98	0.98	0.98
Risk aversion	γ	10	29	29	29
Labor adjusted Risk aversion		10	10	10	10
Intertemporal elasticity of substitution	ψ	2.0	2.0	2.0	2.0
Production function/Aggregator parameters					
Capital share	α	0.3	0.3	0.3	0.3
Depreciation rate of physical capital	δ_K	11%	11%	11%	11%
Depreciation rate of intangible capital	δ_S	11%	11%	11%	11%
Weight on physical investment	ν	0.88	0.93	0.85	0.85
Elasticity of substitution	η	2.50	3.8	1.60	1.60
Adjustment Costs	ξ	5	5	5	5
TFP parameters					
Average growth rate	μ	2.0%	2.0%	2.0%	2.0%
Volatility of short-run risk	σ_a	5.08%	4.40%	5.38%	4.98%
Volatility of long-run risk	σ_x	0.86%	0.75%	0.64	0.75
Autocorrelation of expected growth	ρ	0.925	0.925	0.925	0.925
Time-0 Risk exposure of new investment	ϕ_0	0	0	0	0
Time-1 Risk exposure of new investment	ϕ_1	1	1	0	0.5

Notes - This table reports the parameter values used for our calibrations referring to the model extensions studied in section V. All models are calibrated at an annual frequency. The labor adjusted risk aversion is computed as γ/α (see Swanson (2012)). Extension 1 features adjustment costs on intangible investment. In Extension 2 we also add endogenous labor. In Extension 3 and 4 we retain adjustment costs and endogenous labor and set ϕ_1 to 0 and 0.5, respectively, to study 2-period transitions of productivity exposure.

amount of blueprints in period t; therefore in each period we sample from the distribution (C.5) a number of independent draws of $\theta_{i,t}$ proportional to S_t. We compute the market-to-book ratio for all new production units which are constructed from the implemented blueprints. This procedure gives a panel of market-to-book ratios of newly established production units which can be used to plot the empirical density. In Figure C.2 , we compare this density (denoted by circles) to the observed empirical distribution (denoted by dots) of the market-to-book ratio of new IPO firms in the SDC Platinum data set. Our sample ranges from 1970 to 2009 and includes 44,922 firms. Figure C.2 suggests that our choice of the $G(I, S)$ function conforms well with the microeconomic evidence on the cross-sectional distribution of market-to-book ratio of new IPO firms.