Investment and CEO compensation under limited commitment ☆

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Abstract

We extend the neoclassical investment model (Hayashi (1982)) to allow for limited commitment on compensation contracts. We consider three types of limited commitment: i) managers cannot commit to compensation contracts that provide lower continuation utility than their outside options; ii) shareholders cannot commit to negative net present value projects; iii) both the managers and the shareholders cannot commit. We characterize the optimal contract under general convex adjustment cost functions and provide examples for which closed-form solutions can be obtained. We show that, as in the data, small firms invest more, grow faster, and have a higher Tobin's Q than large firms under the optimal contract. In addition, the pattern of the dependence of CEO compensation on past performance implied by our model is also consistent with empirical evidence.

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1. Introduction

This paper develops a tractable continuous-time framework that incorporates limited commitment of financial contracts into the neoclassical investment model. We consider an environment in which a risk-neutral shareholder owns an investment project but does not have access to the production technology and has to delegate investment decisions to a risk-averse manager. We study three types of limited commitment. First, the manager cannot commit to

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compensation contracts that provide lower continuation utility than their outside options. We call this limited commitment on the manager side. Second, the shareholder cannot commit to negative net present value (NPV) projects. The second type will be referred to as limited commitment on the shareholder side. In the third case, we consider limited commitment on both the shareholder side and the manager side, or two-sided limited commitment. We show that the optimal contract can be characterized by an ordinary differential equation (ODE), and different types of limited commitment boil down to restrictions on the boundary conditions of the ODE. We provide a regulated Brownian-motion-based characterization of the optimal compensation contract, and we analyze firms' optimal investment policies under all three types of limited commitment.

Our model is consistent with several stylized facts on firm investment and CEO compensation. First, we show that under limited commitment, small firms invest more, pay fewer dividends, and grow faster than large firms.

Despite the constant returns to scale technology, small firms in our model invest more because managers in small firms are poorly diversified and growing large improves risk sharing. The constant returns to scale technology allows our model to generate a power law in the firm size distribution, and agency frictions are responsible for the dependence of investment and payout on firm size. In our model, wage contracts are a form of operating leverage. Because managers are risk averse and shareholders are risk neutral, optimal risk sharing requires managerial compensation to stay constant whenever the limited commitment constraint does not bind. However, since shareholders cannot commit to negative NPV projects, as the firm value approaches zero, adverse productivity shocks must be accompanied by reductions in managerial compensation so that the firm value stays nonnegative at all times. A binding shareholder-side limited commitment constraint, which is more likely to happen in young and small firms, limits risk sharing and reduces efficiency. As a result, limited commitment on the shareholder side gives rise to an additional marginal benefit of investment in young and small firms: investment and growth alleviate the agency problem and improve risk sharing.

The fact that small firms invest more, pay fewer dividends, and grow faster than larger firms is well documented in the literature. Evans (1987) and Hall (1987) show that small firms grow faster than large firms. Small firms are less likely to pay out dividends, as documented by Fama and French (2001), among others. Gala and Julio (2011) find that firm size is a robust predictor of investment rates even after controlling for many other variables, such as Tobin's Q and firm cash flow.

Some previous models with limited commitment are also consistent with the fact that small firms invest more and grow faster (For example, Albuquerque and Hopenhayn (2004)). Our model and the Albuquerque and Hopenhayn (2004) model, however, have several main differences. First, they assume risk-neutral managers and consider limited commitment on the manager side only. Our model allows for risk aversion and two-sided limited commitment. Second, their model relies on a decreasing returns to scale technology. Small firms grow faster in their model because capital is more productive; that is, the marginal physical benefit of investment is higher. Our model features constant returns to scale. Small firms grow faster because growth mitigates the agency problem; that is, the marginal agency benefit of investment is higher. Third, because Albuquerque and Hopenhayn (2004) assume a decreasing returns to scale technology and stationary productivity shocks, firms eventually reach their optimal size and no long-run growth occurs. Our model generates long-run growth and is consistent with fat tails in the firm size distribution as in Luttmer (2007). In addition, because of the decreasing returns to scale technology and the identical discount rates of shareholders and managers, firms eventually grow out of the limited commitment constraint in Albuquerque and Hopenhayn (2004). In our model with two-sided limited commitment, the limited commitment constraint binds in the long run.

Second, our model is also consistent with another stylized fact on firm investment and CEO performance. Under the optimal contact with limited commitment, CEO compensation is history dependent. In particular, limited commitment on the manager (shareholder) side implies that CEO compensation is an increasing function of the historical highest (lowest) level of firm size even after controlling for the current size of the firm.

Our model with one-sided limited commitment implies that compensation depends on the best historical performance of the firm. This implication is similar to that in the classic paper of Harris and Holmstrom (1982). Its empirical support is well documented in labor economics, for example, Beaudry and DiNardo (1991) and McDonald and Worswick (1999). Consistent with previous literature, using the EXECUCOMP database in COMPUSTAT, we show that CEO compensation increases with the best historical performance of the firm even after controlling for current performance.

Our calibrated model also features limited commitment on the shareholder side. We show that in this case, CEO compensation depends not only on the historical best performance but also on the historical worst performance of the firm. Consider a firm whose value is driven toward zero by a sequence of negative productivity shocks. Because shareholders cannot commit to negative NPV projects, they optimally reduce CEO compensation to keep the firm value nonnegative. At the same time, optimal risk sharing requires that CEO pay stays constant unless the limited commitment constraint binds. As a result, subsequent positive shocks do not affect CEO compensation. In this case, CEO compensation is determined by the historical worst performance of the firm where the current level of compensation is set.

The above feature distinguishes our model from those with one-sided limited commitment in the previous literature. In Harris and Holmstrom (1982), managerial compensation responds to positive productivity shocks but is downward rigid. In a consumption risk-sharing context, Krueger and Perri (2006) obtain similar results. Krueger and Perri (2006) also show that in the data, consumption responds to both positive and negative productivity shocks. In our model with two-sided limited commitment, managerial pay responds to both positive and negative productivity shocks. We confirm this implication of our model by using EXECUCOMP data. In particular, we show that CEO compensation is positively correlated with not only the best historical performance of the firm, but also the worst historical performance of the firm, even after controlling for the current size of the firm.

Third, limited commitment on the manager side implies that small firms have a higher Tobin's Q than large firms. The negative relationship between Tobin's Q and firm size is well documented in the literature. In our model, investment is efficient and long-run growth is optimal under the first best. Limited commitment lowers the marginal benefit of investment in large and mature firms and encourages investment in young and small firms. As a result, small firms have a higher valuation ratio than large firms.

We characterize the optimal dynamic contract for general convex adjustment cost functions and provide a closed-form solution for a special case. The closed-form solution allows us to derive an explicit expression of the marginal agency cost of investment and relate it to the fundamental parameters of the model, such as the manager's outside option, the market interest rate, and the volatility of the project.

Our paper belongs to the large literature that studies the implications of dynamic agency for firms' investment decisions and financing policies. See, for example, Quadrini (2004), Clementi and Hopenhayn (2006), and DeMarzo and Fishman (2007a). Within this literature, our paper is more closely related to those papers that emphasize the role of limited commitment.[1] Albuquerque and Hopenhayn (2004) provide a general framework to study firms' financing and investment decisions under limited commitment. Lorenzoni and Walentin (2007) focus on the relationship between Tobin's Q and investment. Schmid (2012) studies the quantitative implications of limited commitment on firm financing and investment decisions in a neoclassical model. Rampini and Viswanathan (2010, 2013) study the implications of limited commitment on firms' risk management and capital structure decisions. All of the above-mentioned papers consider limited commitment on the manager side, whereas our framework allows for two-sided limited commitment. Ai et al. (2013) consider a general equilibrium model with two-sided limited commitment. Different from Ai et al. (2013), this paper focuses on the analysis of the optimal contract design problem and its implications for the Q-theory of investment and for the history dependence of compensation contracts. Two recent papers study limited commitment in the context of human capital accumulation. Zhang (2014) focuses on its effect on the volatility of cash flow and the volatility of wage. Bolton et al. (2014) analyze the implications of limited commitment on firms' risk management and liquidity management policies.

On a methodological level, this paper builds on recent developments in the literature on continuous-time dynamic contracting. DeMarzo and Sannikov (2006) consider a principal-agent problem with cash flow diversion and risk-neutral agents. Biais et al. (2010) focus on the case in which the agent controls the arriving rates of large but infrequent events. Biais et al. (2013) provide an excellent survey and synthesis on the literature of dynamic moral hazard and firm financing.[2]

[1] The large literature on limited commitment of financial contracts includes, for example, Kehoe and Levine (1993), Kocherlakota (1996), and Quadrini and Marimon (2011).

[2] See also Tchistyi (2006), DeMarzo and Fishman (2007b), He (2009), Piskorski and Tchistyi (2010), Nikolov and Schmid (2012), Zhu (2013), and Ai and Li (2012).

DeMarzo et al. (2009) also study a dynamic agency problem in a neoclassical investment model. They consider a cash diversion problem with risk-neutral agents. Different from DeMarzo et al. (2009), our focus is on limited commitment, and we allow for risk-averse agents. Grochulski and Zhang (2011) consider a risk-sharing problem with limited commitment. Miao and Zhang (2013) develop a duality-based method to solve limited commitment problems. Neither Grochulski and Zhang (2011) nor Miao and Zhang (2013) focus on firms' investment decisions and the implications of limited commitment for the Q-theory of investment.

The rest of this paper is organized as follows. We describe the model setup in Section 2. We derive the optimal contract under various types of limited commitment frictions and provide an example with a closed-form solution in Section 3. We calibrate our model and evaluate its quantitative implications on CEO pay and firm investment in Section 4. Section 5 concludes.

2. Model setup

A risk-neutral principal owns a project that produces output from capital using a constant returns to scale technology: $Y_t = F(K_t, X_t)$, where $F(\cdot)$ is a concave and constant returns to scale production function, K_t is the capital stock at time t, and X_t is a vector of inputs that can be purchased in a competitive market. Let W_t be the vector of input prices, and assume there is no aggregate uncertainty so that $W_t = W$ is time invariant. In this case, the operating profit can be written as

$$\Pi_t = \max_{X_t} \{F(K_t, X_t) - WX_t\} = zK_t, \tag{1}$$

where z is the marginal product of capital in equilibrium. Here, operating profit is linear in K_t because competitive factor markets equalize the marginal product of capital across firms, as in Hayashi (1982).

The capital accumulation technology is given by

$$dK_t = (I_t - \delta K_t)\,dt + K_t \sigma dB_t, \tag{2}$$

where $\{B_t\}_{t=0}^{\infty}$ is a standard Brownian motion, $\delta > 0$ is the depreciation rate of capital, and I_t is the investment at time t.

The principal cannot operate the technology and must hire a risk-averse manager to do it. Let

C_t denote the compensation of the manager at time t. The manager's preference is represented by constant relative risk aversion (CRRA) utility:

$$E\left[\int_0^\infty e^{-\beta t}\frac{1}{1-\gamma}C_t^{1-\gamma}dt\right], \tag{3}$$

where $\beta > 0$ is the discount rate and $\gamma > 0$ is the coefficient of relative risk aversion of the manager.

The principal's objective is to maximize the net present value of the project,

$$E\left[\int_0^\infty e^{-rt}\left(\Pi_t - H\left(I_t, K_t\right) - C_t\right)dt\right], \tag{4}$$

subject to the participation constraint of the agent,

$$\left\{E\left[\int_0^\infty e^{-\beta t}C_t^{1-\gamma}dt\right]\right\}^{\frac{1}{1-\gamma}} \geq U. \tag{5}$$

In equation (4), $r > 0$ is the risk-free interest rate, and $H\left(I_t, K_t\right)$ is an increasing, convex, and constant returns to scale adjustment cost function. For simplicity, we focus on the case in which $r = \beta$ in this paper. That is, we assume that the principal and the agent have the same discount rates.[3] Equation (5) is the participation constraint of the manager. We assume that the manager's reservation utility is U. Therefore, the contract has to provide at least life-time utility U to the manager at the outset.

In this paper, we consider two types of limited commitment. The first type of limited commitment is that the manager cannot commit to compensation contracts that yield continuation utility that is lower than that provided by outside options. Similar to the model of Albuquerque and Hopenhayn (2004), we assume that the manager can always take away a fraction θ of the firm's total capital and default on the contract.[4] Upon default, the manager may use the capital that he absconds with to produce consumption goods, but he is permanently excluded from participating in any form of financial market. This assumption, together with the constant returns to scale technology, implies that the manager's outside option is proportional to the size of the firm K_t, which we denote as $\bar{u}\left(\theta\right)K_t$, or $\bar{u}K_t$ for simplicity. The functional form of $\bar{u}\left(\theta\right)$ is given in Appendix

[3]Extensions of the model that include different discount rates are straightforward and available upon request. Allowing for different discount rates does not change the basic properties of the optimal contract.

[4]This assumption is also related to the limited commitment model of Kehoe and Levine (1993) and Kiyotaki and Moore (1997).

Appendix A of the paper. Because the manager cannot commit to contracts that yield continuation utility lower than that provided by outside options, $\bar{u}(\theta) K_t$, the manager's utility under the optimal contract must be higher than the outside option at all times:

$$\left\{ E_s \left[\int_s^{\infty} e^{-\beta(t-s)} C_t^{1-\gamma} dt \right] \right\}^{\frac{1}{1-\gamma}} \geq \bar{u}(\theta) K_s, \quad for\ all\ s > 0, \tag{6}$$

with $E_s[\cdot]$ being the conditional expectation operator based on time s information.

The second type of limited commitment we consider is that the shareholder cannot commit to negative NPV projects. In this case, efficiency requires that the continuation value of the firm remain nonnegative at all times:

$$E_s \left[\int_s^{\infty} e^{-r(t-s)} \left(\Pi_t - H(I_t, K_t) - C_t \right) dt \right] \geq 0, \quad for \quad all \quad s > 0. \tag{7}$$

3. Managerial compensation

In this section, we consider the optimal contract design problem with various forms of limited commitment, namely, the optimal investment problem in which the principal maximizes the total value of the firm (4), subject to the participation constraint (5) and the limited commitment constraint, (6) and/or (7).

3.1. Assumptions

To guarantee a well-behaved solution, we make additional assumptions about the preference of the manager and the production and capital accumulation technology. First, we assume that the production technology is efficient enough so that disinvestment is never optimal in the first-best case. This is Assumption 1.

Assumption 1.

$$z > \beta + \delta \tag{8}$$

Next, we impose some regularity conditions on the adjustment cost function. Given that $H(I, K)$ is constant returns to scale, we can write, without loss of generality,

$$H(I, K) = h\left(\frac{I}{K}\right) K$$

for some increasing and convex function $h(\cdot)$. We impose the following conditions on the $h(i)$ function.

Assumption 2. *The function $h(i)$ is twice continuously differentiable and $h'(i), h''(i) > 0$ for all i. In addition,*

$$h(0) = 0, \quad h'(0) = 1, \tag{9}$$

and

$$i_Z < r + \delta, \tag{10}$$

where i_Z is the unique solution to $h(i) = z$.

Intuitively, condition (9) implies that the adjustment cost is low enough at $I = 0$. Together with Assumption 1, it guarantees that the optimal level of investment is strictly positive in the first-best case. Note that this assumption includes the standard quadratic adjustment cost as a special case. Condition (10) implies that the adjustment cost increases fast enough with investment so that the firm value remains finite in the first-best case. Note that i_Z is the level of investment that exhausts all output. Under condition (10), firms' maximum sustainable growth rate, i_Z, is strictly less than the discount rate.[5]

Finally, we impose the following joint assumptions on technology and preference to guarantee that the manager's investment is always positive and that utility is always finite upon default.

Assumption 3. *If $\gamma < 1$, then*

$$\frac{1}{2}\sigma^2 < \frac{1}{\gamma}\left[\delta + \frac{1}{1-\gamma}(z - \beta)\right]. \tag{11}$$

Alternatively, if $\gamma > 1$, then

$$\frac{1}{2}\sigma^2 < \frac{1}{\gamma}\left(\frac{1}{\gamma - 1}\beta - \delta\right). \tag{12}$$

Upon default, managers are not allowed to participate in any risk-sharing contract, and we need to make sure that their continuation utility remains well defined. In the case $\gamma < 1$, the manager's utility is always finite under Assumption 1. However, if the discount rate is too high or the volatility of the technology is too low, the investment will be negative. Inequality (11) ensures that managers do not have any incentive to disinvest after default. When $\gamma > 1$, the discounted utility of the manager may not be finite. Condition (12) provides an upper bound on the volatility of the technology relative to the discount rate that is sufficient for the finiteness of the manager's

[5] Assumption 2 is a sufficient condition that guarantees the finiteness of the firm value in the first-best case as well as in the cases with agency frictions. If this assumption is violated, the firm value is not defined in the first-best case, but may still be finite under agency frictions. We maintain Assumption 2 in Sections 3-5, but relax this assumption in our calibration in Section 6.

utility upon default.

3.2. Optimal compensation policy

We study managerial compensation policy in this section. Under the optimal dynamic contract, managerial compensation exhibits a very simple form of history dependence. We show that in the model with limited commitment on the manager side, compensation is an increasing function of the manager's best historical performance, and in the case in which shareholders cannot commit to negative NPV projects, compensation increases with the worst historical performance of the firm. In Section 4, we show that both features of our calibrated model are consistent with empirical evidence on CEO compensation.

As is standard in the literature on dynamic contracting, we consider the recursive formulation of the optimal contacting problem. We define the continuation utility of the manager at time t as follows:

$$U_t = \left\{ E_t \left[\int_t^{\infty} e^{-r(s-t)} C_s^{1-\gamma} ds \right] \right\}^{\frac{1}{1-\gamma}} . \tag{13}$$

By using promised utility U and total capital stock K as state variables, we can formulate the optimal contract design problem as a dynamic programming problem. We denote $V(U,K)$ as the shareholder's value function and $C(U,K)$, $I(U,K)$ as the compensation and investment policy functions. Under our normalization, the manager's utility is homogeneous of degree one, and the production technology is constant returns to scale. Therefore, homogeneity of the contracting problem implies that the value function and policy functions are homogeneous of degree one in K. That is, there exist functions $v(\cdot)$, $c(\cdot)$, and $i(\cdot)$ such that

$$V(U,K) = v(u)K; \quad C(U,K) = c(u)K; \quad I(U,K) = i(u)K, \tag{14}$$

where $u = \frac{U}{K}$ is the continuation utility normalized by the capital stock.

To understand the normalized compensation policy function, $c(u)$, we use equation (14) to write $\ln c(u_t) = \ln C_t - \ln K_t$. Also, homogeneity implies that the limited commitment constraint (6) can be written as

$$u_t \geq \bar{u}. \tag{15}$$

That is, the normalized continuation utility of the manager has to be higher than the fraction of the firm's assets that he can abscond with upon default. Because $c(u)$ is strictly increasing (which we will prove formally in Appendix Appendix B), inequality (15) is equivalent to $\ln c(u_t) \geq \ln c(\bar{u})$ for all t.

We make two observations here. First, because the principal is risk neutral and the manager is risk averse, optimal risk sharing requires that compensation stays constant unless a limited commitment constraint binds. That is, whenever the limited commitment constraints do not bind, C_t is constant, and movements in $\ln c(u_t)$ are completely driven by changes in $\ln K_t$. Second, managerial compensation, C_t has to change to satisfy the limited commitment constraints whenever they are binding.

Together, the above two observations imply that in the case of limited commitment on the manager side, $\ln c(u_t) = \ln C_0 - \ln K_t + l_t$, where l_t must be zero if the limited commitment constraint does not bind, and l_t must increase when the constraint binds so that $\ln c(u_t) \geq \ln c(\bar{u})$ is satisfied at all times. Formally, l_t is a regulator (Harrison (1985)) that keeps the process $\ln C_0 - \ln K_t$ above $\ln c(\bar{u})$ for all t.

Likewise, in the case of limited commitment on the shareholder side, let $\hat{u}$ be such that $v(\hat{u}) = 0$. The shareholder's limited commitment constraint (7) is equivalent to $u_t \leq \hat{u}$ for all t. A similar argument suggests that $\ln c(u_t)$ can be obtained from $\ln C_0 - \ln K_t$ by imposing a regulator at the upper barrier, $\ln c(\hat{u})$. These characterizations are summarized by the following proposition.

Proposition 1. (Optimal Compensation under Limited Commitment)

1. *In the case of limited commitment on the manager side, for all $t \geq 0$,*

$$\ln c(u_t) = \ln C_0 - \ln K_t + l_t, \tag{16}$$

   where $\{l_t\}_{t=0}^{\infty}$ is the minimum increasing process such that $l_0 = 0$ and $\ln c(u_t) \geq \ln c(\bar{u})$ for all $t \geq 0$.

2. *In the case of limited commitment on the shareholder side, there exists a $\hat{u} > 0$ such that for all $t \geq 0$,*

$$\ln c(u_t) = \ln C_0 - \ln K_t - m_t, \tag{17}$$

   where $\{m_t\}_{t=0}^{\infty}$ is the minimum increasing process such that $m_0 = 0$ and $\ln c(u_t) \leq \ln c(\hat{u})$ for all $t \geq 0$.

3. *In the case of two-sided limited commitment, there exists a $\tilde{u} > 0$ such that for all $t \geq 0$,*

$$\ln c(u_t) = \ln C_0 - \ln K_t + l_t - m_t, \tag{18}$$

   where $\{l_t\}_{t=0}^{\infty}$, $\{m_t\}_{t=0}^{\infty}$ are the minimum increasing processes, such that $l_0 = m_0 = 0$ and $\ln c(\bar{u}) \leq \ln c(u_t) \leq \ln c(\tilde{u})$ for all $t \geq 0$.

Proof. See Appendix Appendix B. ■

The above proposition formalizes the notion that the optimal compensation is constant whenever neither of the limited commitment constraints is binding, makes minimum adjustment to keep the firm value nonnegative whenever the limited commitment constraint on the shareholder side binds, and makes minimal adjustment to keep the manager from taking his outside option whenever the limited commitment constraint on the manager side binds.

In dynamic agency problems, the dependence of the optimal contract on history is typically complicated and summarized by state variables including continuation utility U_t. In our model, history dependence takes a simple form. Consider first the case of limited commitment on the manager side. Using the result from Harrison (1985), the regulator can be constructed as follows:

$$l_t = \max\left\{\ln c\left(\bar{u}\right) - \ln C_0 + \ln K_t^*,\ 0\right\},$$

where $K_t^* = \sup_{0 \le s \le t} K_s$ is the running maximum of firm size up to time t. In this case,

$$\ln C_t = \ln C_0 + l_t = \max\left\{\ln c\left(\bar{u}\right) + \ln K_t^*,\ \ln C_0\right\},$$

or

$$C_t = \max\left\{c\left(\bar{u}\right) K_t^*,\ C_0\right\}. \tag{19}$$

That is, managerial compensation is a linear function of the running maximum of firm size. This characterization is consistent with the classic result of Harris and Holmstrom (1982).[6]

Intuitively, because the manager's outside option increases with the size of the firm, compensation must increase to retain the manager whenever K_t increases above some threshold K^* so that the manager's outside option is higher than the present value of the current compensation contract. At this point, the level of compensation is reset to match the manager's outside option $\bar{u}K^*$. Optimal risk sharing implies that further changes in K_t should not affect compensation unless K_t increases above K^*, in which case the current compensation contract needs to be readjusted to match the new high of the outside option. In this case, the historical maximum of K_t is sufficient to determine the level of CEO compensation, as shown in (19).

[6]See also Lustig et al. (2011) and Grochulski and Zhang (2011).

Downward rigidity is a robust feature of the optimal contract under limited commitment on the agents' side. However, as we show in Section 4 of the paper, CEO compensation responds not only to positive but also to negative productivity shocks. Our model with limited commitment on the shareholder side creates a mechanism through which the manager's compensation may decline with negative productivity shocks. By a similar argument as above, in the case of limited commitment on the shareholder side only, the optimal compensation can be written as

$$C_t = \min\left\{c\left(\hat{u}\right)\hat{K}_t,\ C_0\right\}, \tag{20}$$

where $\hat{K}_t = \inf_{0\leq s\leq t} K_s$ is the running minimum of the firm's capital stock up to time t. In this case, managerial compensation is upward rigid but decreases with negative productivity shocks when the firm value hits zero, namely, when the limited commitment constraint on the shareholder side binds.

In the general case of two-sided limited commitment, the optimal compensation-to-firm-size ratio can be represented by a two-sided regulated Brownian motion. In this case, managerial compensation is neither downward rigid nor upward rigid. It stays constant whenever none of the limited commitment constraints binds, increases to prevent the manager from taking the outside option when the limited commitment constraint on the manager side binds, and decreases to keep the firm value from being negative whenever the constraint on the shareholder side binds.

Neither a historical maximum nor a minimum is sufficient to determine CEO compensation in the model with two-sided limited commitment; however, equations (19) and (20) suggest that, quantitatively, CEO compensation should be positively correlated with both the historical high and the historical low of firm size. We confirm this in our calibrated model in Section 4, where we also provide empirical support for this feature of our model.

4. Investment and Tobin's Q

In this section, we study the optimal investment policy under limited commitment. As a benchmark, we start with the first-best case without any agency frictions.

4.1. First best

In the first-best case, the shareholder maximizes the firm value (4) subject to the constraint (5). The optimization problem can be solved in two steps. First, choose the optimal investment policy to maximize the total value of cash flow, $E\left[\int_0^\infty e^{-rt}\left(Y_t - H\left(I_t, K_t\right)\right)dt\right]$. Second, choose the optimal compensation to minimize the cost, $E\left[\int_0^\infty e^{-rt}C_t dt\right]$, subject to the constraint (5). The solution to the optimal contracting problem in the case of the first best is given by the following proposition.

Proposition 2. (The First-Best Case)
The value function of the firm with initial capital stock K and promised utility U is given by

$$V\left(K, U\right) = \bar{v}K - \frac{1}{r}U, \tag{21}$$

where the constant $\bar{v} = h'\left(\hat{\imath}\right)$ and $\hat{\imath}$ is the optimal investment-to-capital ratio given by

$$\hat{\imath} = \arg\max_i \frac{z - h\left(i\right)}{r + \delta - i}. \tag{22}$$

Proof. See Appendix Appendix C. ■

Equation (21) has an intuitive interpretation. The term $\bar{v}K = h'\left(\hat{\imath}\right)K$ is the firm value in the neoclassical model with capital adjustment costs (for example, Hayashi (1982)), and $\frac{1}{r}U$ is the present value of the manager's compensation. Perfect risk sharing implies a constant managerial compensation, $C_t = U$, for all t; therefore, the present value of managerial compensation is simply given by Gordon (1959)'s formula.[7] By the above proposition, the normalized value function is linear:

$$v\left(u\right) = \bar{v} - \frac{1}{r}u. \tag{23}$$

The normalized compensation and investment policies satisfy

$$c\left(u\right)K = U; \quad i\left(u\right) = \hat{\imath} \ \ for \ \ all \ \ u \in \left(0, \infty\right). \tag{24}$$

That is, CEO compensation is constant and does not depend on firm size. The investment rate (investment-to-capital ratio) is also constant over time.

4.2. The HJB equation

Here, we derive the Hamilton-Jacobi-Bellman (HJB) equation that characterizes the value function and study the dynamic of continuation utility, u_t. We show that u_t is negatively correlated with firm size under the optimal contract. This subsection is technical in nature. Readers who are

[7] Note that under our normalization of the utility function, utility and consumption are measured in the same units.

primarily interested in the implications of the model can skip to subsection 4.4 without breaking the flow of the paper.

As in Sannikov (2008), using the martingale representation theorem, the law of motion of promised utility can be written as

$$dU_t = U_t \left\{ \left[\frac{r}{1-\gamma} \left(1 - \left(\frac{C_t}{U_t} \right)^{1-\gamma} \right) + \frac{1}{2}\gamma g_t^2 \sigma^2 \right] dt + g_t \sigma dB_t \right\}, ^{8} \tag{25}$$

where g_t is the sensitivity of continuation utility with respect to the Brownian motion dB_t. As a result, the law of motion of normalized utility, u_t, is given by

$$du_t = u_t \left\{ \mu_u(u_t)\, dt + \left[g(u_t) - 1 \right] \sigma dB_t \right\}, \tag{26}$$

where we denote

$$\mu_u(u) = \frac{r}{1-\gamma} \left(1 - \left(\frac{c(u)}{u} \right)^{1-\gamma} \right) - \left(i(u) - \delta \right) + \left(1 - g(u) + \frac{1}{2} [\text{illegible}] (u) \sigma^2 \right),$$

and $c(u)$, $i(u)$, and $g(u)$ are policy functions.

The normalized value function must satisfy the following ODE, which can be derived from the HJB equation for the dynamic programming problem:

$$0 = \max_{c,i,g} \left\{ \begin{array}{c} \left[z - c - h(i) \right] + v(u) \left[i - r - \delta \right] \\ +uv'(u) \left[\frac{r}{1-\gamma} \left(1 - \left(\frac{c}{u} \right)^{1-\gamma} \right) - (i - \delta) + \frac{1}{2}\gamma g^2 \sigma^2 \right] \\ +\frac{1}{2} u^2 v''(u) (g-1)^2 \sigma^2 \end{array} \right\}. \tag{27}$$

The following lemma summarizes the above result.

Lemma 1. (The HJB Equation)
The normalized value function and policy functions under all three cases of limited commitment must satisfy the above ODE in its domain. The boundary conditions are given by the following:

1. *In the case of limited commitment on the manager side, $v(u)$ is the solution to (27) on $[\bar{u}, \infty)$ with the boundary conditions $v''(\bar{u}) = -\infty$ and $\lim_{u \to \infty} \left[v(u) - \left(\bar{v} - \frac{1}{r} u \right) \right] = 0$.*[9]
2. *In the case of limited commitment on the shareholder side, $v(u)$ is the solution to (27) on $(0, \hat{u}]$, with the boundary conditions $v(\hat{u}) = 0$, $v''(\hat{u}) = -\infty$, and $\lim_{u \to 0} v(u) = \bar{v}$.*
3. *In the case of two-sided limited commitment, $u \in [\bar{u}, \tilde{u}]$, where $v''(u) = -\infty$ for $u = \bar{u}, \tilde{u}$, and $v(\tilde{u}) = 0$.*

[8]Note that to maintain the homogeneity of the problem, we use a monotonic transformation of the expected utility representation of the preference, as in equation (13).

[9]To save notation, we denote the normalized value function for all three types of limited commitment as $v(u)$.

Proof. See Appendix Appendix C. ■

The above lemma provides a characterization of the normalized value function under limited commitment. In the case of limited commitment on the manager side, the outside option provides normalized utility $\bar{u}$. As a result, the normalized utility provided by the compensation contract must be higher than $\bar{u}$ at all times. Due to the limited commitment constraint, (6), the value function in this case must be below the value function for the first-best case in its entire domain. As u becomes large, the probability of the constraint being binding before time T vanishes for any finite T. As a result, the value function converges to the first-best level.

Next, consider the case of limited commitment on the shareholder side. Note that the optimal compensation policy for the first-best case, $C_t = U$ for all t, yields negative firm value because negative shocks reduce the firm's capital stock. That is, $V(K,U) = \bar{v}K - \frac{1}{r}U < 0$ for K small enough. In the case in which the shareholder cannot commit to negative NPV projects, the first-best compensation policy is no longer feasible. Proposition 1 implies that there exists a $\hat{u} > 0$ such that $u_t = \frac{U_t}{K_t} \leq \hat{u}$ for all t under the optimal contract, and $v(u) \geq 0$ for all $u \in (0, \hat{u}]$. As $u \to 0$, the probability of hitting a binding constraint vanishes and the value function converges to the first-best case.

Finally, in the case of two-sided limited commitment, u_t must stay inside the closed interval, $[\bar{u}, \tilde{u}]$, where $v(\tilde{u}) = 0$. Intuitively, two-sided limited commitment reduces the set of feasible payoffs under the optimal contract. As a result, the value function under two-sided limited commitment is strictly below that for the first-best case, and $\tilde{u} < \hat{u}$, where $\hat{u}$ is the highest possible normalized utility for the manager under the optimal contract with shareholder-side limited commitment.

In Figure D.1, we plot the normalized value functions for the first-best case and for all three cases of limited commitment. Note that the normalized value function for the first-best case (solid line) is linear, and as $u \to \infty$, $v(u)$ becomes negative. The value function for the case of limited commitment on the manager side (dashed line) stays below the value function for the first-best case and converges to the former as $u \to \infty$. Also, $u \geq \bar{u}$ in this case, which prevents the manager from defaulting on the contract. The value function for the case of limited commitment on the shareholder side (dash-dotted line) stays below the value function for the first-best case and converges to the former as $u \to 0$. Note that under the optimal contract, $u \leq \hat{u}$ and $v(u) \geq 0$ for all u in its domain when shareholders cannot commit to negative NPV projects. Finally, the value function for the two-sided limited commitment case (dotted line) stays strictly below the first-best case and the cases with one-sided limited commitment. The fact that $\tilde{u} < \hat{u}$ indicates that lack of commitment on the manager side limits the set of feasible continuation utilities that can be supported by the optimal contract.

The dynamics of investment and CEO compensation depend on the dynamics of normalized continuation utility, u. In the first-best case, compensation and continuation utility are constant. As a result, $u_t = \frac{U}{K_t}$ is decreasing in firm size K_t. This is the implication of optimal risk sharing: continuation utility responds less to productivity shocks than it does to output. This basic property is preserved under all three cases of limited commitment. The following lemma characterizes the dynamics of normalized utility under the optimal contract.

Lemma 2. (Dynamics of the Normalized Utility)

The optimal sensitivity of continuation utility, $g(u)$, satisfies $0 < g(u) < 1$ in the interior of its domain. In addition:

1. *In the case of limited commitment on the manager side only,* $\lim_{u\to\bar{u}} g(u) = 1$ *and* $\lim_{u\to\infty} g(u) = 0$.
2. *In the case of limited commitment on the shareholder side only,* $\lim_{u\to\hat{u}} g(u) = 1$ *and* $\lim_{u\to 0} g(u) = 0$.
3. *In the case of two-sided limited commitment,* $\lim_{u\to\bar{u}} g(u) = \lim_{u\to\tilde{u}} g(u) = 1$.

Proof. See Appendix Appendix C. ■

Note that $g(u)$ is the sensitivity of the continuation utility, U_t, with respect to productivity shocks. Perfect risk sharing implies that $g(u) = 0$ and complete autarky is associated with $g(u) = 1$. In general, limited commitment allows partial risk sharing and $0 \leq g(u) \leq 1$. Because the normalized continuation utility is defined as $u_t = \frac{U}{K_t}$, the sensitivity of u_t with respect to productivity shocks is $g(u) - 1$ (see equation (26)), and complete risk sharing as in the first-best case is associated with perfect negative correlation of u_t with respect to productivity shocks.

In the case of limited commitment on the manager side, on the one hand, as $u \to \infty$, the probability of hitting a binding constraint vanishes and $g(u) \to 0$, the first-best level. On the other hand, as normalized utility approaches the binding limited commitment constraint, $\bar{u}$, further increases in K_t are more likely to be associated with increases in compensation to prevent the manager from taking the outside option. That is, continuation utility U_t must increase with K_t. As a result, $g(u)$ increases and approaches 1 at $\bar{u}$. At $\bar{u}$, $\mu_u(u) > 0$ and $g(u) = 1$.

The dynamics of u_t are similar in other cases of limited commitment. In Figure 3, we plot the drift and diffusion coefficient of $\ln u_t$. At $\bar{u}$, the drift is strictly positive and the diffusion coefficient approaches zero, indicating that normalized utility, u_t, reverts back to the interior with probability one. In the interior of $(\bar{u}, \tilde{u})$, $g(u) < 1$ and the diffusion of $\ln u$, $g(u) - 1$ is negative, which is consistent with imperfect risk sharing. As u_t approaches $\tilde{u}$, where the limited commitment constraint on the shareholder side binds, $g(u)$ increases and approaches one, and the drift of $\ln u$ becomes positive, implying that risk sharing is poor in this region, and u_t reverts back to the interior with probability one once it hits the right boundary, $\tilde{u}$.

Note that $g(u) \leq 1$ implies that innovations in u_t are negatively correlated with innovations in K_t. Therefore, firms with large u are more likely to be small in size.[10]This feature is important in understanding the investment-size relationship in our model, which we discuss in the following subsection.

4.3. Optimal investment policy

Empirical evidence suggests that small firms invest more (Gala and Julio (2011)), grow faster (Evans (1987) and Hall (1987)), and pay fewer dividends (Fama and French (2001)). Qualitatively, all three models with limited commitment are consistent with this feature of the data. In the case of limited commitment on the shareholder side, when K_t is low and the firm value is close to zero, shareholders optimally cut managerial compensation to keep the firm value nonnegative. This result implies that managers are poorly diversified in small firms, in which equity value is low. As a result, it is optimal for small firms to cut dividends so that they can increase investment above the first-best level in order to reduce the agency costs associated with lack of risk sharing. In the model with limited commitment on the manager side, because managers' outside options are increasing in firm size, it is optimal for large firms, where managers' outside options are more attractive, to reduce investment in order to keep managers from defaulting.

Formally, firms' investment rate is a function of the normalized utility, $\frac{I_t}{K_t} = i(u_t)$, which we characterize in the following proposition.

Proposition 3. (Optimal Investment)

[10]Note, however, that the relationship between K and u is not one-to-one.

1. *Under all three types of limited commitment, the optimal investment rate,* $i(u)$*, is an increasing function of normalized continuation utility,* u*.*
2. *Let* $\hat{\imath}$ *be the investment rate under the first best. Then, under limited commitment on the manager side,*
$$i(u) \leq \hat{\imath} \text{ for all } u \in [\bar{u}, \infty), \text{ and } \lim_{u \to \infty} i(u) = \hat{\imath}. \tag{28}$$
3. *Under limited commitment on the shareholder side,*
$$i(u) \geq \hat{\imath} \text{ for all } u \in (0, \hat{u}], \text{ and } \lim_{u \to 0} i(u) = \hat{\imath}. \tag{29}$$

Proof. See Appendix Appendix C. ■

Note that $u_t = \frac{U_t}{K_t}$. Optimal risk sharing implies that managers' continuation utility, U_t, responds less to productivity shocks than it does to K_t. Therefore, under the optimal contract, u_t and firm size, K_t, are negatively correlated. (This observation is consistent with the observation $g(u) < 1$ made in the last subsection.) The fact that $i(u)$ is an increasing function implies that the investment rate and firm size are negatively correlated.

We plot the optimal investment policies as functions of normalized utility in Figure D.3. The optimal investment rate in the first-best case (solid line) is constant at $\hat{\imath}$. Firms under-invest when managers cannot commit (dashed line). In this case, because managers' outside options are increasing in firm size, it is optimal to reduce investment to prevent the limited commitment constraint from binding as normalized utility gets close to $\bar{u}$. As $u \to \infty$, the probability of hitting a binding constraint vanishes, and investment approaches the first-best level. On the other hand, firms overinvest when shareholders cannot commit (dash-dotted line). In this case, because small firms are more likely to experience a binding commitment constraint, which is associated with imperfect risk sharing, limited commitment creates additional incentives for investment. As a result, firms close to $\hat{u}$ invest at a level well above the first-best optimum, $\hat{\imath}$. As $u \to 0$, the probability of hitting a binding constraint vanishes, and firms' investment approaches the first-best level. Finally, two-sided limited commitment produces underinvestment for firms with low levels of u (mostly large firms) and overinvestment for firms with high levels of u (typically small firms). This is shown as the dotted line in Figure D.3. As a result, in the case with two-sided limited commitment, small firms are likely to overinvest relative to the first-best level, and large firms are likely to underinvest.

To summarize, risk sharing implies that normalized continuation utility is negatively correlated with firm size. Limited commitment implies that the firm's investment rate is increasing in normalized utility. Together they imply that small firms invest more and grow faster, despite the constant returns to scale technology. We explore these features of our model quantitatively in Section 5 of the paper.

4.4. An example with a closed-form solution

In this section, we provide an example in which the optimal contract with limited commitment can be solved in closed-form. For simplicity, we consider only the case with limited commitment on the manager side.[11] The closed-form solution allows us to derive an intuitive expression for the marginal agency cost of investment and relate it to the preference and technology parameters of the model. We also provide an explicit expression for Tobin's Q and establish an inverse relationship between Tobin's Q and firm size.

We assume that the adjustment cost function $H(I,K)$ takes the following simple form:

$$H(I,K) = \begin{cases} I & if \quad \frac{I}{K} \in [0,\hat{\imath}] \\ \infty & if \quad \frac{I}{K} \notin [0,\hat{\imath}] \end{cases} . \tag{30}$$

That is, the investment-to-capital ratio has an upper and a lower bound. The per unit cost of investment is one as long as investment is nonnegative and $\frac{I}{K}$ stays below $\hat{\imath}$. Note that $\hat{\imath}$ is the maximum possible investment rate.[12]

To make the problem interesting, we first make some assumptions about the parameters of the model. The first assumption guarantees that investment is efficient and that the firm value is finite in the first-best case.

Assumption 4. $z > \delta + r > \hat{\imath}$

Intuitively, z is the marginal product of capital, and $r + \delta$ is the user's cost of capital (Jorgenson (1963)). The first part of Assumption 1, $z > \delta + r$, implies that the marginal benefit of investment exceeds its marginal cost in the first-best case. In this case, the optimal approach is to invest at the maximum level, $\hat{\imath}$. The second part of the assumption imposes an upper bound on the highest possible investment rate, which guarantees that the firm value is finite in the first-best case.

[11] Closed-form solutions for other cases can be obtained in a similar way.

[12] Our use of the notation $\hat{\imath}$ is consistent with that in previous sections. As we will see, under our assumptions, $\hat{\imath}$ is the optimal level of investment in the first-best case.

The second assumption guarantees that the lifetime utility of the agent is always finite under the optimal contract.

Assumption 5. $\gamma < \max\left\{1,\ \frac{2(\hat{\imath}-\delta+r)}{\sigma^2}\right\}$

Our last assumption ensures that the agency cost is high enough so that under the optimal contract, investment is lower than the first-best case with nonzero probability.

Assumption 6.

$$\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} < 1 + \frac{1}{r}\frac{\gamma}{\alpha_1 - 1}\left(\frac{\alpha_1+\gamma-1}{\alpha_1}\right)^{\frac{\gamma}{1-\gamma}}\bar{u}, \tag{31}$$

where $\alpha_1$is given by equation (D.9) in Appendix Appendix D.

The term $\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}}$ is the marginal benefit of investment under the first-best case. Under Assumption 4, $\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} > 1$ and it is optimal to invest at the highest possible level without any agency friction. As we show later, the second term on the right-hand side of (31) can be interpreted as the maximum level of the marginal agency cost of investment. Under Assumption 6, the marginal benefit of investment is lower than the maximum marginal cost of investment. As a result, investment is lower than its first-best level, at least in some states of the world.

Let $V_K(U,K)$ denote the partial derivative of firms' value function with respect to K. By homogeneity, $V_K(U,K)$ is only a function of the normalized utility, u. In Appendix Appendix C, we show that $V_K(U,K)$ can be written as

$$V_K(U,K) = \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} - \chi(u),$$

where $\chi(u)$, which is given in equation (D.21) in Appendix Appendix D, can be interpreted as the marginal agency cost of investment. Because the physical cost of investment always equals one, the optimal investment policy is a bang-bang solution: invest at the maximum level if $\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} > 1+\chi(u)$ and invest at the minimum level if $\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} < 1+\chi(u)$. As we show in Appendix Appendix C, $\chi(u)$ is monotonically decreasing and there exists a $u^* \in (\bar{u},\infty)$ such that $\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} > 1+\chi(u)$ if and only if $u > u^*$.

We plot the marginal cost of investment as a function of the normalized utility, u, in the top panel of Figure D.4. The bottom panel of the same figure is the optimal investment policy. Intuitively, when $\frac{U_t}{K_t} \le u^*$, the manager's continuation utility is small relative to the size of the firm. In this case, more investment is likely to increase the manager's outside option and therefore his incentive to default in the near future. As shown in the figure, the marginal agency cost of investment is high and exceeds the marginal benefit (dotted line). Therefore, not investing is optimal in this region. In the region where $u \geq u^*$, default is less likely and the marginal agency cost of investment is low. Therefore, the optimal choice is to invest at the maximum level, $\hat{\imath}$. Clearly, the marginal agency cost of investment $\chi(u)$ is a decreasing function of u and is maximized at $\bar{u}$, where the limited commitment constraint binds. Intuitively, small values of u imply that managers' continuation utility is low relative to their outside options and that the limited commitment constraint is more likely to bind in the near future.

The special form of investment policy, along with the regulated Brownian motion characterization of compensation policy, allows us to provide a closed-form solution to the contracting problem, which we summarize in the following proposition.

Proposition 4. *(Closed-Form Solution)*
There exists a $u^* > \bar{u}$ such that for any $u \in [\bar{u}, \infty)$, the normalized compensation policy $c(u)$ is implicitly defined by

$$u^{1-\gamma} = \begin{cases} c(u)^{1-\gamma} + A_1 c(u)^{1-\gamma-\beta_1} + A_2 c(u)^{1-\gamma+\beta_2} & \text{if } u \in [\bar{u}, u^*) \\ c(u)^{1-\gamma} + B_1 c(u)^{1-\gamma-\alpha_1} & \text{if } u \in [u^*, \infty) \end{cases} . \tag{32}$$

The normalized value function is given by

$$v(u) = \begin{cases} \frac{z}{r+\delta} - \frac{1}{r}c(u) + C_1 c(u)^{1-\beta_1} + C_2 c(u)^{1+\beta_2} & \text{if } u \in [\bar{u}, u^*) \\ \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} - \frac{1}{r}c(u) + D_1 c(u)^{1-\alpha_1} & \text{if } u \in [u^*, \infty) \end{cases} , \tag{33}$$

where the constants A_1, A_2, B_1, C_1, C_2, D_1, and u^* are determined by a set of value matching and smooth pasting conditions listed in Appendix Appendix D.
Proof. *See Appendix Appendix D.* ■

We denote $\bar{V}(U, K)$ to be the total value of the firm, including the value of the cash flow to the shareholder and the manager. That is, for a project with initial condition (U, K),

$$\bar{V}(U, K) = E\left[\int_0^\infty e^{-rt}(zK_t - H(I_t, K_t))\,dt\right].$$

Due to homogeneity, the average Q, or the ratio of the total market value of the firm to its total capital stock, $\frac{\bar{V}(U,K)}{K}$, is only a function of normalized utility, u. We denote

$$q_A(u) = \frac{\bar{V}(U, K)}{K}$$

to be the Tobin's Q of the firm. The closed-form solution for $v(u)$ can be used to derive a closed form solution for Tobin's Q:

$$q_A(u) = \begin{cases} \frac{z}{r+\delta} + C_3 c(u)^{1-\alpha_1} + C_4 c(u)^{1+\alpha_2} & \text{if } u \in [\bar{u}, u^*) \\ q_A(u) = \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} + D_3 c(u)^{1-\beta_1} & \text{if } u \in [u^*, \infty) \end{cases}, \tag{34}$$

where the constants C_3, C_4, and D_3 are determined by a set of value matching and smooth pasting conditions listed in Appendix Appendix D. In the same appendix, we show that $q(u)$ is an increasing function of the normalized utility u. Intuitively, as u increases, the probability of an immediate binding constraint becomes smaller and investment is more efficient, and as a result, the valuation ratio increases. In addition, as $u \to \infty$, $q_A(u)$ converges to the first-best level, $\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}}$.

Because normalized utility is negatively correlated with firm size (Lemma 2), our model is consistent with the empirical fact that firm size is negatively correlated with Tobin's Q. Figure D.5 provides a useful way in which to visualize the monotonic relationship between firm size and Tobin's Q. Consider a firm with initial condition (U_0, K_0), and assume $\frac{U_0}{K_0} > \bar{u}$. The initial managerial compensation is $c(u_0) K_0$. The limited commitment constraint binds for the first time at time τ when $c(\bar{u}) K_\tau = c(u_0) K_0$, that is, when K_t increases and reaches $\frac{c(u_0)}{c(\bar{u})} K_0$. Before this time, there is a one-to-one relationship between K_t and u_t; therefore, investment and Tobin's Q are only functions of firm size. This allows us to plot Tobin's Q (top panel) and investment (bottom panel) as a function of firm size in Figure D.5. In the top panel, Tobin's Q is monotonically decreasing in firm size. It reaches its minimum level at K^*, where the manager's limited commitment constraint starts to bind. As the firm size becomes smaller, the limited commitment constraint is less likely to bind, and Tobin's Q increases and converges to its first-best level (dashed line). The bottom panel of Figure D.5 plots the investment policy as a function of firm size. It is clear that small firms invest at a higher level and large firms at a lower level.

The maximum marginal agency cost of investment $\chi(\bar{u})$ is an important determinant of investment behavior. If $1 + \chi(\bar{u}) < \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}}$, then limited commitment does not distort investment. Otherwise, limited commitment lowers investment for firms close to the binding constraint. To emphasize the dependence of the maximum marginal agency cost of investment on the fundamental parameters of the model, we denote it as $\chi(\bar{u}; r, \hat{\imath}, \sigma^2)$. The following proposition relates the marginal agency cost of investment to the fundamental parameters of the model.

Proposition 5. (Marginal Agency Cost of Investment)

The maximum marginal agency cost of investment, $\chi\left(\bar{u}; r, \hat{\imath}, \sigma^2\right)$, is strictly increasing in $\bar{u}$, strictly increasing in $\hat{\imath}$ and σ^2, and strictly decreasing in the interest rate, r.

Proof. *See Appendix D.* ■

The above proposition is intuitive. A higher $\bar{u}$ increases managers' outside options and exacerbates the commitment problem. As a result, the marginal agency cost of investment increases. A large $\hat{\imath}$ is associated with a high growth rate of the firm. Because the manager's outside option is a linear function of firm size, a higher growth rate implies that the limited commitment constraint is more likely to bind in the near future. The interest rate has the opposite effect as the growth rate. Agency cost arises because of the possibility of a binding limited commitment constraint in the future. A higher interest rate is associated with heavier discounting of future cash flows and therefore lower agency cost in terms of present values. Finally, the marginal agency cost of investment increases with volatility. Limited commitment is welfare reducing because it prevents perfect risk sharing. A higher volatility implies a higher cost of imperfect risk sharing and increases the agency cost of investment.

5. Calibration

In this section, we calibrate our model and evaluate its quantitative implications for investment and CEO compensation. We choose the interest rate $r = 4\%$ to match the average return on government bonds and equities in the United States during the postwar period. We choose the capital depreciation rate to be 11%, consistent with the standard real business cycle literature. To guarantee a stationary distribution of firm size, we assume that firms die at a constant rate $\kappa = 10\%$, which is consistent with the same moment used in the firm dynamics literature, for example, Luttmer (2007). We use the standard quadratic adjustment cost function,

$$H(I, K) = I + \frac{1}{2}\phi\left(\frac{I}{K} - i^*\right)^2 K,$$

and set $\phi = 1$. We choose $i^* = 21\%$ to be the steady-state investment rate. We calibrate $z = 28\%$ to match the average log growth rate of firms in the COMPUSTAT data set, 10%. We choose $\sigma = 40\%$ to match the volatility of sales growth rate for COMPUSTAT firms. We simulate a cross section of $20,000$ firms for 350 years and remove the first 300 years of the simulated data to eliminate the effect of initial conditions and to focus on the steady state implied by the model.

Thanks to the constant returns to scale technology, our model generates a power law in firm size, which we plot in Figure D.6. Recall that a random variable X follows a power law or a Pareto distribution if for all x,

$$P(X > x) \propto x^{-\xi}, \tag{35}$$

where ξ is the power law coefficient. Note that (35) can be written as $\ln P(X > x) \propto -\xi \ln x$. In Figure D.6, we plot $\ln P(K > x)$ against $\ln x$ in the data and in our model, where K stands for firm size. The circles, plus signs, and stars are the distributions of firm size reported by the Small Business Administration for the years 1992, 2000, and 2006, respectively. The solid line is the distribution of firm size generated by our calibrated model. Our model produces a power law coefficient of 1.04, close to the estimate of 1.06 reported in Luttmer (2007).[13]

Despite the constant returns to scale technology, small firms invest more, grow faster, and have a higher Tobin's Q in our model because of agency frictions. We plot the average growth rate of firms in Figure D.7. We sort all firms in our COMPUSTAT data set into ten groups by size and plot the average growth rate of firms in each group as a function of firm size. We use two measures of firm size, the total capital stock of the firm (circles) and the total value of the firm's asset including the total market value of equity and the total book value of debt (stars). The decrease in growth rates with respect to firm size is robust for both measures. In the same figure, we also plot firm growth rates as a function of size generated by our calibrated model as the squares. Our model closely matches the negative relationship between firm size and growth rates in the data.

In Figure D.8, we plot Tobin's Q as a function of firm size in the data and in the model. In the data, we sort firms by size (as measured by the total capital stock of the firm) into ten groups. We follow Erickson and Whited (2000) in constructing Tobin's Q. The book value of a firm is measured by its total asset (at) and the market value of the firm is measured by the total market value of equity plus the total book value of debt less deferred taxes and investment tax credit (txditc). To mitigate the effect of measurement error, we compute a three-year moving average of Tobin's Q for each firm and plot the average Tobin's Q for all groups. This is shown in the top panel of Figure

[13] Firm size is measured by the total number of employees in the data. In our model, labor and capital are equivalent measures of firm size. The constant returns to scale technology and a frictionless labor market imply that capital and labor are proportional at the firm level.

D.8. In the model, because we have a large enough number of firms in our simulation, we simply plot the average Tobin's Q for each group, which is depicted in the bottom panel of the figure. Our model closely matches the decreasing pattern of Tobin's Q with respect to size, although the magnitude of the decrease is somewhat lower in the model.

The limited commitment on the manager side is responsible for the inverse relationship between firm size and Tobin's Q. As we showed earlier, the lack of commitment on the manager side mainly distorts the investment policies of large firms, where managers have more incentive to default. Because investment is less efficient in large firms, the Tobin's Q in large firms is lower compared with small ones.[14]

The classical Q-theory relationship fails in our model because agency frictions generate a wedge between marginal Q, the marginal benefit of investment, and the average Q, which is the ratio of the market value to the total capital stock of the firm. In Figure D.9, we plot the marginal Q as a function of Tobin's average Q implied by our model. Note that the marginal Q is increasing in the average Q, but the relationship is highly nonlinear.

In our model, investment is correlated with cash flow even after controlling for Tobin's Q. We compare the following investment to cash flow sensitivity regression in the data and in our model:

$$\frac{I_t}{K_t} = \xi_0 + \xi_Q Q_t + \xi_{CF} \frac{CF_t}{K_t} + \varepsilon_t, \tag{36}$$

where I_t and CF_t are the total investment and cash flow between period t and period $t + 1$, and K_t and Q_t are the total capital stock and the Tobin's Q (the ratio of market value to book value) of the firm at the beginning of period t. In the data, we use all publicly traded companies in the COMPUSTAT data base for the period 1950-2009. We measure firm investment by its total capital expenditure (capxv) minus sales of property, plant, and equipment (sppe). We measure cash flow, CF_t, as total sales (sales) minus cost of goods sold (cogs). As before, we measure capital stock

[14] For the same reason, limited commitment on the shareholder side lowers Tobin's Q in small firms with little impact on large firms. Under two-sided limited commitment, the relationship between Tobin's Q and firm size depends on which effect is stronger. Quantitatively, the effect of limited commitment on the manager side typically dominates if the model is calibrated to match the average growth rate of firms in the data. The reason is that this constraint is much more likely to bind than limited commitment on the shareholder side. In the data, firms on average grow at about 10% per year, and small firms grow even faster. As a result, firms typically grow quickly out of the limited commitment constraint on the shareholder side. However, the limited commitment constraint on the manager side binds infinitely often in the long run, because as firms grow, managers' outside options increase as well.

as the total book value of assets (at) minus current assets (act). This allows us to exclude cash and other liquid assets that may not be appropriate components of physical capital. Finally, our construction of Tobin's Q follows Erickson and Whited (2000).

We report the results of these regressions in Table 1. As in the data, the regression coefficient on cash flow is significantly positive even after controlling for Tobin's Q. This implication of our model is similar to that in Lorenzoni and Walentin (2007). In our model, the above regression rejects the hypothesis that investment does not depend on cash flow after controlling for Tobin's Q for two reasons. The first reason is nonlinearity. As shown in Figure D.9, marginal Q, which determines investment, is a nonlinear function of Tobin's (average) Q. Regression (36) does not account for this nonlinear relationship. The second reason is time aggregation. In our continuous-time model, marginal Q perfectly predicts investment during an infinitesimally small interval. However, the discrete time regression (36) does not account for changes in Q within a period. In the model, productivity shocks arrive continuously and move investment, cash flow, and Q infinitely often within a small time interval. An econometrician who observes marginal Q continuously will be able to perfectly predict investment. However, Q samples taken at discrete time points are not sufficient statistics of investment and cannot drive out all the positive covariance between cash flow and investment.

Because our model has only one source of productivity shocks, the regression coefficients and the R^2 of the regression in our model are much larger than those in the data. Because we have a large number of firms, the T-statistics of all coefficients in our model are also very large.

As we show in Section 3 of the paper, managerial compensation is an increasing function of both the historical best and the historical worst performance of the firm. We test these implications of the model in the data. We consider the following regression:

$$C_t = \xi_0 + \xi_C Current\ Size + \xi_{MAX} Historical\ Max + \xi_{MIN} Historical\ Min + \varepsilon_t. \tag{37}$$

In the above equation, C_t stands for the level of CEO compensation in the current period. *Current Size* is the current size of the firm, *Historical Max* is the best historical performance of the firm, and *Historical Min* is the worst historical performance of the firm.

We report our regression results in Table 2. In the data, we measure CEO compensation by the sum of the total compensation including option grants (TDC1) of the top three executives of the firm reported in the EXECUCOMP data base. Firm performance is measured by the sum of the total market value of the firm's equity and the total book value of the firm's debt. We measure the best historical performance of the firm by the highest historical value of the firm in the last ten years and measure the historical worst performance of the firm by the lowest historical value of the firm in the last ten years. Our sample include all firms jointly listed in the EXECUCOMP and COMPUSTAT data base from 1992-2009 for which the above variables can be constructed. Our regression include 16759 firm-year observations in total. We follow the same procedure in the regression with data generated by our simulated model. Because our model is a stationary one without aggregate growth, we detrend the variables in the data by the average growth rate of the U.S. economy during the postwar period, 2% per year. In the data, CEO compensation is increasing in the historical best performance of the firm, and the regression coefficient is highly significant. This evidence is consistent with that obtained in the labor economics literature in the context of wage dynamics; see, for example, Beaudry and DiNardo (1991) and McDonald and Worswick (1999). As in the data, our model produces a highly significant regression coefficient on the historical maximum size of the firm with a similar magnitude. In addition, as in the data, CEO compensation in our model is increasing with the historical worst performance of the firm, with a less significant regression coefficient. Note that in the model, CEO compensation is adjusted according to the historical maximum of firm size after a sequence of positive productivity shocks raise managers' outside options and force shareholders to increase their compensation levels to retain the manager. CEO compensation is modified to reflect the historical minimum level of capital stock after the firm's equity value reaches zero and the shareholders' limited commitment constraint starts to bind. In the model, the regression coefficient on $Historical\ Max$ is less significant because firms on average are growing, and the limited commitment constraint on the shareholder side is much less likely to bind than that on the manager side.

6. Conclusion

In this paper, we incorporate limited commitment into the neoclasscial investment model and study its implications for investment and managerial compensation. We characterize the optimal contract for general adjustment cost functions and provide closed-form solutions for special cases.

The constant returns to scale technology allows our model to match the power law of the distribution of firm size, and limited commitment generates the negative relationship between Tobin's Q and firm size. In addition, we show that limited commitment on the manager side produces a monotonic relationship between CEO compensation and his best historical performance, and limited commitment on the shareholder side implies that CEO compensation is increasing in the firm's worst historical performance. We show that both features of the model are consistent with the empirical evidence.

Appendix

Appendix A. Manager's Outside Option, $\bar{u}(\theta)$

We first consider managers' utility maximization problem upon default. Given an initial level of capital stock K_0, the manager maximizes his utility by making the following optimal consumption and investment decisions:

$$\max_{\{C_t, I_t\}_{t=0}^{\infty}} \left\{ E\left[\int_0^{\infty} e^{-\beta t} C_t^{1-\gamma} dt\right]\right\}^{\frac{1}{1-\gamma}}$$
$$C_t + H(I_t, K_t) = zK_t$$
$$dK_t = (I_t - \delta K_t)\, dt + K_t \sigma dB_t.$$

We adopt the expected utility representation of the same preference and define $W(K) = E\left[\int_0^{\infty} e^{-\beta t} \frac{1}{1-\gamma} C_t^{1-\gamma} dt\right]$. Homogeneity of the above optimization problem implies that $W(K) = \frac{1}{1-\gamma}\omega K^{1-\gamma}$ for some ω, and the optimal investment-to-capital ratio is a constant, which we denote as i^*. The constants ω and i^* are given by the following lemma.

Lemma 3. (Manager's Utility Upon Default)
Under Asumptions 1-3, i^* is the unique solution to the following equation on $(0, i_Z)$:

$$h'(i) = \frac{z - h(i)}{\beta - (1-\gamma)(i-\delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2}, \tag{A.1}$$

and

$$\omega = [z - h(i^*)]^{-\gamma} h'(i^*) > 0. \tag{A.2}$$

Proof. *Clearly, $W(K)$ has to satisfy the following HJB equation:*

$$\beta W(K) = \frac{1}{1-\gamma}[zK - H(I,K)]^{1-\gamma} + [I - \delta K] W_K + \frac{1}{2} K^2 \sigma^2 W_{KK}.$$

Using the fact that $W(K) = \frac{1}{1-\gamma}\omega K^{1-\gamma}$, the above can be simplified to

$$\beta\omega = [z - h(i)]^{1-\gamma} + (1-\gamma)(i-\delta)\omega - \frac{1}{2}\gamma(1-\gamma)\sigma^2\omega. \tag{A.3}$$

If the optimal investment is interior, it has to satisfy the first order condition:

$$[z - h(i)]^{-\gamma} h'(i) = \omega. \tag{A.4}$$

Equation (A.3) and (A.4) jointly determine ω and i^*.

We can obtain (A.1) by substituting ω from (A.4). Define

$$f(i) = z - h(i) - h'(i)\left[\beta - (1-\gamma)(i-\delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right]. \tag{A.5}$$

To prove the lemma, it is enough to show that equation (A.5) has a unique solution on $(0, i_Z)$. Note

$$f'(i) = -\left\{\gamma h'(i) + h''(i)\left[\beta - (1-\gamma)(i-\delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right]\right\}.$$

Below we show $f'(i) < 0$ on $(0, i_Z)$, and $f(0) < 0$ and $f(i_Z) > 0$.

Consider first the case $\gamma < 1$. It is straightforward to show that in this case $f'(i) < 0$ on $(0, i_Z)$. Also,

$$f(0) = z - \left[\beta + (1-\gamma)\delta + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right] > 0$$

under condition (11) in Assumption 3. In addition,

$$f(i_Z) = -h'(i_Z)\left[\beta - (1-\gamma)(i_Z - \delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right] < 0, \tag{A.6}$$

because

$$\begin{aligned} & \beta - (1-\gamma)(i_Z - \delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2 \\ > \; & \beta - (1-\gamma)\beta + \frac{1}{2}\gamma(1-\gamma)\sigma^2 \\ = \; & \gamma\beta + \frac{1}{2}\gamma(1-\gamma)\sigma^2 > 0, \end{aligned}$$

under condition (10) in Assumption 2.

Next, consider the case $\gamma > 1$. To show that $f'(i) < 0$ on $(0, i_Z)$, we prove that $\beta - (1-\gamma)(i-\delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2 > 0$ on $(0, i_Z)$. Given $\gamma > 1$, it is enough to show

$$\beta + (1-\gamma)\delta + \frac{1}{2}\gamma(1-\gamma)\sigma^2 > 0, \tag{A.7}$$

which is true under condition (12). Also, $f(0) = z - \left[\beta + (1-\gamma)\delta + \frac{1}{2}\gamma(1-\gamma)\sigma^2\right] > 0$ given Assumption 1 and $\gamma > 1$. Using the equality in (A.6), to show $f(i_Z) < 0$, it is enough to prove

$$\beta - (1-\gamma)(i_Z - \delta) + \frac{1}{2}\gamma(1-\gamma)\sigma^2 > 0,$$

which is implied by inequality (A.7).

Finally, it is straightforward to verify the sufficiency of the first order condition (A.4) for optimality due to the concavity of the objective function. This completes the proof. ■

Note that upon default, the manager abscond with a fraction θ of the capital stock of the firm. Therefore, $\bar{u}(\theta)K = [(1-\gamma)W(\theta K)]^{\frac{1}{1-\gamma}} = \omega^{\frac{1}{1-\gamma}}\theta K$. Hence

$$\bar{u}(\theta) = \omega^{\frac{1}{1-\gamma}}\theta,$$

where ω is given by (A.2) in Lemma 3.

Appendix B. Optimal Compensation Policy

We first establish the following lemma.

Lemma 4. (Monotonicity and Concavity)

The value function $V(U, K)$ is strictly decreasing in U and concave in K.

Proof. We prove this for the case of two-sided limited commitment. The cases with one-sided limited commitment can be established by the same argument.

To see that $V(U,K)$ is strictly decreasing in U, consider $U_1 > U_2$, and $U_j \in (\bar{u}K, \tilde{u}K)$, for $j = 1, 2$. Let $\left\{\hat{C}_{1,t}, \hat{I}_{1,t}\right\}_{t=0}^{\infty}$ be the optimal policy with initial conditions (U_1, K). We establish $V(U_1, K) < V(U_2, K)$ by constructing a policy $\left\{\tilde{C}_{2,t}, \tilde{I}_{2,t}\right\}_{t=0}^{\infty}$ such that:

1. It satisfies constraints (5), (6) and (7) under initial condition (U_2, K).
2. The alternative policy yields higher firm value, that is,

$$E\left[\int_0^{\infty} e^{-rt}\left(z\tilde{K}_t - H\left(\tilde{I}_{2,t}, \tilde{K}_t\right) - \tilde{C}_{2,t}\right) dt\right] > V(U_1, K). \tag{B.1}$$

The policy $\left\{\tilde{C}_{2,t}, \tilde{I}_{2,t}\right\}_{t=0}^{\infty}$ can be constructed as follows. Define a policy $\{C'_t, I'_t\}_{t=0}^{\infty}$ as:

$$C'_t = \frac{U_2}{U_1}\hat{C}_{1,t};\ I'_t = \hat{I}_{1,t}\ for\ \ all\ \ t \geq 0,$$

Let $\tilde{\tau}$ be the stopping time such that the one of limited commitment constraint binds for the first time under the policy $\{C'_t, I'_t\}_{t=0}^{\infty}$. Then $\left\{\tilde{C}_{2,t}, \tilde{I}_{2,t}\right\}_{t=0}^{\infty}$ is given by:

$$\tilde{C}_{2,t} = \left\{\begin{array}{ll} C'_t & for\ t \leq \tilde{\tau} \\ \hat{C}_{1,t} & for\ t > \tilde{\tau} \end{array}\right. .$$

$$\tilde{I}_{2,t} = \hat{I}_{1,t}\ \ for\ \ all\ t \geq 0.$$

One can verify that the proposed policy is feasible under (U_2, K) and yields strictly higher firm value than $\left\{\hat{C}_{1,t}, \hat{I}_{1,t}\right\}_{t=0}^{\infty}$. This proves monotonicity.

To see $V(U,K)$ is concave in K, consider K_1, K_2 such that $U \in [\bar{u}K_j, \tilde{u}K_j]$, for $j = 1, 2$. We need to show $V\left(U, K^{\lambda}\right) \geq \lambda V(U, K_2) + (1-\lambda) V(U, K_2)$, where $\lambda \in [0,1]$ and

$$K^{\lambda} = \lambda K_1 + (1-\lambda) K_2. \tag{B.2}$$

Let $\{C_{j,t}, I_{j,t}\}_{t=0}^{\infty}$ be an optimal policy with initial condition (U, K_j), for $j = 1, 2$, respectively. Let

$$C_{\lambda,t} = \lambda C_{1,t} + (1-\lambda) C_{2,t}, \tag{B.3}$$

and

$$I_{\lambda,t} = \lambda I_{1,t} + (1-\lambda) I_{2,t}, \tag{B.4}$$

for all t. It is enough to show that $\{C_{\lambda,t}, I_{\lambda,t}\}_{t=0}^{\infty}$ is feasible under the initial condition $\left(U, K^{\lambda}\right)$.

For a fixed sample path of shocks, $\{B_t\}_{t=0}^{\infty}$, let $K_{j,t}$ be the sample path of capital stock under initial conditions K_j, and policies $\{I_{j,t}\}_{t=0}^{\infty}$, for $j = 1, 2$, and λ, respectively. That is,

$$dK_{j,t} = (I_{j,t} - \delta K_{j,t})\, dt + K_{j,t} dB_t, \quad for\ j = 1, 2. \tag{B.5}$$

Note that given equations (B.2) and (B.4), the above equation implies $K_{\lambda,t} = \lambda K_{1,t} + (1-\lambda) K_{2,t}$ for all t. Therefore, condition (5) and (6) are satisfied because of the concavity of the utility function, and condition (7) is satisfied because the adjustment cost function $H(I,K)$ is convex. This establishes the concavity of the value function. ■

Lemma 5. (Constant Payment to the Agent)

Suppose the limited commitment constraints (6) and (7) do not bind during the time interval (t_1, t_2)*. Then,* $C_t = C_{t'}$ *for all* $t, t' \in (t_1, t_2)$*.*

Proof.

Here, we only consider the case of two-sided limited commitment. The other two cases can be established in a similar way.

In the dynamic programming formulation of the optimal contracting problem, firms maximize profit (equation (4)) subject to the law of motion of state variables, (U, K) (equations (25) and (2), respectively), and the limited commitment constraints, (6) and (7).

For any $U \in (\bar{u}K, \tilde{u}K)$, the limited commitment constraints do not bind and the value function $V(U, K)$ must satisfy the following HJB equation:

$$rV(U,K) = \max_{C,i,h} \left\{ \begin{array}{c} zK - h(i)K - C + V_K(U,K)K[i-\delta] + \frac{1}{2}V_{KK}(U,K)K^2\sigma^2 \\ +V_U(U,K)U\left[\frac{r}{1-\gamma}\left(1-\left(\frac{C}{U}\right)^{1-\gamma}\right) + \frac{1}{2}\gamma h^2\sigma^2\right] \\ +\frac{1}{2}V_{UU}(U,K)U^2h^2\sigma^2 + V_{K,U}(U,K)KUh\sigma^2 \end{array} \right\}. \tag{B.6}$$

To prove Lemma 5, it is enough to show that $dC(U,K) = 0$ for $U \in (\bar{u}K, \tilde{u}K)$, where $C(U,K)$ is the optimal compensation policy. The first order condition with respect to C gives:

$$rU^\gamma V_U(U,K) = \frac{1}{C^{-\gamma}}. \tag{B.7}$$

The first order condition with respect to h gives:

$$[\gamma V_U(U,K) + UV_{UU}(U,K)]h + KV_{K,U}(U,K) = 0. \tag{B.8}$$

By Equation (B.7), to prove $dC(U,K) = 0$, it is enough to show $d[U^\gamma V_U(U,K)] = 0$.

Using Ito's lemma, the diffusion term of $d[U^\gamma V_U(U,K)]$ is

$$\begin{aligned} & \frac{\partial}{\partial U}[U^\gamma V_U(U,K)]Uh\sigma + \frac{\partial}{\partial K}[U^\gamma V_U(U,K)]K\sigma \\ = \; & U^\gamma[(\gamma V_U(U,K) + UV_{UU}(U,K))h + KV_{K,U}(U,K)]\sigma, \end{aligned}$$

which equals 0 by condition (B.8).

The drift term of $d[U^\gamma V_U(U,K)]$ is

$$\begin{aligned} & \frac{\partial}{\partial U}[U^\gamma V_U(U,K)]U\left[\frac{r}{1-\gamma}\left(1-\left(\frac{C}{U}\right)^{1-\gamma}\right) + \frac{1}{2}\gamma h^2\sigma^2\right] \\ & +\frac{\partial}{\partial K}[U^\gamma V_U(U,K)]K[i-\delta] + \frac{1}{2}\frac{\partial^2}{\partial U^2}[U^\gamma V_U(U,K)]U^2h^2\sigma^2 \\ & +\frac{1}{2}\frac{\partial^2}{\partial K^2}[U^\gamma V_U(U,K)]K^2\sigma^2 + \frac{\partial^2}{\partial K \partial U}[U^\gamma V_U(U,K)]UKh\sigma^2. \end{aligned} \tag{B.9}$$

The envelope condition for the maximization problem (B.6) implies

$$\begin{aligned}
rV_U(U,K) &= V_{UK}K(i-\delta)+\frac{1}{2}V_{UKK}(U,K)K^2\sigma^2 \\
&+V_{UU}(U,K)U\left[\frac{r}{1-\gamma}\left(1-\left(\frac{C}{U}\right)^{1-\gamma}\right)+\frac{1}{2}\gamma h^2\sigma^2\right] \\
&+V_U(U,K)\frac{\partial}{\partial U}\left[\frac{r}{1-\gamma}U\left(1-\left(\frac{C}{U}\right)^{1-\gamma}\right)+\frac{1}{2}U\gamma h^2\sigma^2\right] \\
&+\frac{1}{2}V_{UUU}(U,K)U^2h^2\sigma^2+V_{UU}(U,K)Uh^2\sigma^2 \\
&+V_{KUU}(U,K)KUh\sigma^2+V_{KU}(U,K)Kh\sigma^2,
\end{aligned}$$

Using (B.8), the above envelope condition implies that

$$\begin{aligned}
&\gamma V_U(U,K)\frac{r}{1-\gamma}\left[1-\left(\frac{C}{U}\right)^{1-\gamma}\right]+V_{UK}(U,K)K(i-\delta)+\frac{1}{2}V_{UKK}(U,K)K^2\sigma^2 \\
&+V_{UU}(U,K)U\left[\frac{r}{1-\gamma}\left(1-\left(\frac{C}{U}\right)^{1-\gamma}\right)+\frac{1}{2}\gamma h^2\sigma^2\right]+\frac{1}{2}V_{UUU}(U,K)U^2h^2\sigma^2 \\
&+h\sigma^2\left[KUV_{KUU}(U,K)-\frac{1}{2}\gamma hV_U(U,K)\right]=0
\end{aligned}$$

One can show that the above equation is equivalent to the drift term of $d\left[U^{\gamma}V_U(U,K)\right]$ in equation (B.9) being zero.

■

Proof of Proposition 1:

By Lemma 4, the value function $V(U,K)$ is strictly decreasing in U and strictly concave in K. This implies that the normalized value function $v(u)$ must be strictly decreasing and concave. The first order condition (B.7) implies that the normalized policy function $c(u)$ must satisfy:

$$c(u)=\left[-\beta v'(u)\right]^{\frac{1}{\gamma}}u. \tag{B.10}$$

Using (B.10), the strict monotonicity and concavity of $v(u)$ can be used to show that the policy function $c(u)$ must be strictly increasing. This implies that the limited commitment constraint on the manager (or respective, shareholder) side binds if and only if $c(u_t)=c(\bar{u})$ $(or c(\tilde{u})$, respectively).

Based on the discussion in the paragraph before Proposition 1, $\ln c(u_{t+\Delta})-\ln c(u_t)=(\ln C_0-\ln K_{t+\Delta})-(\ln C_0-\ln K_t)$ whenever $c(\bar{u})<\ln c(u_t)<c(\tilde{u})$ for Δ small. In addition $c(\bar{u})\leq\ln c(u_t)\leq c(\tilde{u})$ for all t. Together these two conditions imply that $\ln c(u_t)$ can be obtained from $\ln C_0-\ln K_t$ by imposing a two-sided regulator. The construction of the two sided regulator on page 23 of Harrison (1985) can be adapted to show that the definition of $\{l_t\}_{t=0}^{\infty}$ and $\{m_t\}_{t=0}^{\infty}$ in Proposition 1 coincides with the definition of two-sided regulator in Harrison (1985) (that is, condition 1)-3) on page 22).

Appendix C. Optimal Investment Policy

Proof of Proposition 2:

Note that the optimization problem can be solve in two steps. First, choose the optimal investment policy to maximize the total value of the cash flow, $E\left[\int_0^\infty e^{-rt}\left(Y_t - H\left(I_t, K_t\right)\right)dt\right]$. Second, choose the optimal compensation to minimize the cost $E\left[\int_0^\infty e^{-rt}C_t dt\right]$ subject to the constraint (5). Because the agent is risk aversion, the optimal compensation policy must be constant, $C_t = U$, and the present value of managerial compensation is $\frac{1}{r}U$.

Homogeneity of the problem implies that the total present value of the firm's cash flow must be linear in K. Denote $\bar{v}K = E\left[\int_0^\infty e^{-rt}\left(Y_t - H\left(I_t, K_t\right)\right)dt \middle| K_0 = K\right]$, the HJB of the optimal investment problem can be written as:

$$r\bar{v}K = \max_i \left\{\left(z - h\left(i\right)\right)K + K\left(i - \delta\right)\bar{v}\right\}.$$

This implies

$$\bar{v} = \max_i \frac{z - h\left(i\right)}{r + \delta - i}. \tag{C.1}$$

Under Assumptions 1 and 2, one can show that problem (C.1) has a unique maximizer on $(0, i_Z)$ that satisfies the first order condition:

$$f\left(i\right) = z - h\left(i\right) - h'\left(i\right)\left[r + \delta - i\right] = 0.$$

Note that $f\left(i\right) = 0$ has a unique solution on $(0, i_Z)$ because under Assumptions 1 and 2, $f'\left(i\right) < 0$ on $(0, i_Z)$, and $f(0) > 0, f(i_Z) < 0$. This completes the proof of Proposition 2.

We first state the following Lemma.

Lemma 6. Ito's Formula for Regulated Brownian Motion

Let the law of motion of $\{\xi_t\}_{t=0}^\infty$ and $\{K_t\}_{t=0}^\infty$ be given by:

$$d\xi_t = -\ln K_t - dm_t + dl_t \tag{C.2}$$
$$dK_t = K_t\left[\left(i\left(\xi_t\right) - \delta\right)dt + \sigma dB_t\right], \tag{C.3}$$

where m_t and l_t are regulators at the upper barrier, ξ_H and lower barrier, ξ_L such that ξ_t stays within the interval $[\xi_L, \xi_H]$ at all times, and $i\left(\xi\right)$ is a continuous function of ξ. Define

$$W\left(\xi, K\right) = E\left[\int_0^\infty e^{-rt}\eta\left(\xi_t\right)K_t^\rho dt \middle| \xi_0 = \xi, K_0 = K\right], \tag{C.4}$$

then

$$W\left(\xi, K\right) = G\left(\xi\right)K^\rho,$$

where $G\left(\xi\right)$ is second order continuously differentiable and satisfies the following ODE on $[\xi_L, \xi_H]$:

$$0 = \eta\left(\xi\right) + \left[\rho\left(i\left(\xi\right) - \delta\right) + \frac{1}{2}\rho\left(\rho - 1\right)\sigma^2 - r\right]G\left(\xi\right) - \left(i\left(\xi\right) - \delta + \frac{1}{2}\sigma^2\right)G'\left(\xi\right) + \frac{1}{2}\sigma^2 G''\left(\xi\right), \tag{C.5}$$

with boundary conditions $G'\left(\xi_L\right) = G'\left(\xi_H\right) = 0$.

Proof. *Use Proposition (7) on page 84 in Harrison (1985).* ■

Proof of Lemma 1:

Here we only prove the first part of the lemma, i.e. the case with limited commitment on the manager side. Part 2 and 3 of the lemma can be proved using a similar argument.

First, using the homogeneity of value function and policy functions, we can show that the ODE in (B.6) reduces to (27). Therefore, the normalized value function $v(u)$ must satisfy (27) in the interior when the limited commitment constraint does not bind.

We first prove the boundary condition $\lim_{u\to\infty} v(u) - \left[\bar{v} - \frac{1}{r}u\right] = 0$. The fact that $v(u) \leq \bar{v} - \frac{1}{r}u$ is obvious. We show that $\lim_{u\to\infty} v(u) - \left[\bar{v} - \frac{1}{r}u\right] \geq 0$ by constructing a feasible policy that approximate first best firm value in the limit as $u \to \infty$.

Let

$$\tau(u_0) = \inf\left\{t : K_t > \left(\frac{\alpha_1}{\alpha_1 - 1 + \gamma}\right)^{\frac{1}{1-\gamma}} \frac{u_0}{\bar{u}} K_0\right\}, \tag{C.6}$$

where $\alpha_1 > 1$ is given by equation (D.9). Consider the following policy:

$$\tilde{C}_t = \begin{cases} u_0 K_0 & t < \tau(u_0) \\ c(u_t) K_t & t \geq \tau(u_0) \end{cases} ; \; \tilde{I}_t = \begin{cases} \hat{\imath} K_t & t < \tau(u_0) \\ i(u_t) K_t & t \geq \tau(u_0) \end{cases} \tag{C.7}$$

That is, we adopt the first best compensation and investment policy until the stopping time $\tau(u_0)$, and switch to the the optimal policy afterwards.

Denote $\tilde{V}(K_0, u_0 K_0)$ the value of the firm under the policy $\left\{\tilde{C}_t, \tilde{I}_t\right\}_{t=0}^{\infty}$, that is,

$$\tilde{V}(K_0, u_0 K_0) = E\left[\int_0^\infty e^{-rt}\left(zK_t - H\left(\tilde{I}_t, K_t\right) - \tilde{C}_t\right) dt\right].$$

By Lemma 7 below, the policy $\left\{\tilde{C}_t, \tilde{I}_t\right\}_{t=0}^{\infty}$ satisfies constraints (5) and (6). Therefore,

$$\tilde{V}(K_0, u_0 K_0) \leq V(K_0, u_0 K_0). \tag{C.8}$$

In addition, Lemma 7 implies that as $u_0 \to \infty$, $\tau(u_0) \to \infty$. As a result,

$$\tilde{V}(K_0, u_0 K_0) \to \left[\bar{v} - \frac{1}{r}u_0\right] K. \tag{C.9}$$

Together, conditions (C.8) and (C.9) implies $\lim_{u_0\to\infty} V(u_0 K_0, K_0) - \left[\bar{v} - \frac{1}{r}u_0\right] K_0 \geq 0$, as needed.

We next prove the boundary condition $\lim_{u\to\bar{u}} v''(u) = -\infty$. Note that the first order condition (B.10) implies

$$-rv'(u) = \left[\frac{c(u)}{u}\right]^\gamma.$$

Assuming $c(u)$ is differentiable (which we will prove below), we have:

$$-rv''(u) = \gamma\left[\frac{c(u)}{u}\right]^\gamma \left[\frac{c'(u)}{c(u)} - \frac{1}{u}\right].$$

Given $c(\bar{u}), \bar{u} > 0$, to prove $\lim_{u\to\bar{u}} v''(u) = -\infty$, it is enough to show $\lim_{u\to\bar{u}} \frac{c(u)}{u} = \infty$, which we establish below.

We define $\xi = \ln c(u)$. Note that equation (B.10) implies that the policy function $c(u)$ is strictly increasing in u on $[\bar{u}, \infty)$. Therefore, $\ln c(u)$ has an inverse, and (assuming differentiability,)

$$\frac{c'(u)}{c(u)} = \frac{d\xi}{du} = \frac{1}{\frac{du}{d\xi}}. \tag{C.10}$$

Recall that Lemma 5 implies that $\xi_t = \ln c(u_t)$ is a regulated Brownian motion of the form (C.2) and (C.3), where in (C.3), the function $i(\xi)$ is defined as $i(\xi) = i\left(c^{-1}\left(e^{\xi}\right)\right)$, and $\xi_L = \ln c(\bar{u})$, $\xi_H = \infty$.[15] Given the law of motion of (K_t, ξ_t), under the optimal contract, the continuation utility of the manager can be computed from Lemma 6: $U_t = \left[W\left(\xi_t, K_t\right)\right]^{\frac{1}{1-\gamma}}$, where $W(\xi, K)$ is a special case of equation (C.4) with $\eta(\xi) = e^{(1-\gamma)\xi}$. Using Lemma 6, $u_t = \frac{U_t}{K_t} = G\left(\xi_t\right)^{\frac{1}{1-\gamma}}$ is twice continuously differentiable. This proves the differentiability of $c(u)$.

In addition, $\frac{du}{d\xi} = \frac{1}{1-\gamma} G(\xi)^{\frac{\gamma}{1-\gamma}} G'(\xi)$. By Lemma 6, as $\xi \to \xi_L$, $G'(\xi) \to 0$. In fact, it is straightforward to verify $\frac{1}{1-\gamma} G'(\xi) \to 0^+$ using monotonicity. Combine with (C.10), we have:

$$\lim_{u \to \bar{u}} \frac{c'(u)}{c(u)} = \lim_{u \to \bar{u}} \frac{d\xi}{du} = \lim_{\xi \to \ln c(\bar{u})} \frac{1}{\frac{du}{d\xi}} = \infty,$$

as needed.

Note that Lemma **1** does not claim that the solution to HJB (27) with the stated boundary conditions is unique. In fact, boundary condition $\lim_{u \to \bar{u}} v''(u) = -\infty$ typically do not uniquely determine the value function. However, the above arguement implies that if we transform the state variable by using $\xi = \ln c(u)$, then both the normalized value of the firm and the utility of the agent can be represented by functions of ξ, the boundary conditions of which are well defined as stated in Lemma 6. These functions are uniquely determined as the solution to (C.5) with the corresponding boundary conditions. Our construction of the value function in the example with closed form solution follows this procedure.

The proof of the boundary condition $\lim_{u \to \infty} v(u) - \left(\bar{v} - \frac{1}{r} u\right) = 0$ requires the construction of a feasible policy that approximate the first best firm value as $u \to \infty$. The following lemma establishes the feasibility of the policy $\left\{\tilde{C}_t, \tilde{I}_t\right\}_{t=0}^{\infty}$ and the fact that it approximate the first best firm value as $u_0 \to \infty$.

Lemma 7. *The policy, $\left\{\tilde{C}_t, \tilde{I}_t\right\}_{t=0}^{\infty}$ satisfies the participation constraint (5) and the limited commitment constraint on the manager side, (6) under the initial condition, (u_0, K_0). In addition, $\tau(u_0) \to \infty$ with probability one as $u_0 \to \infty$.*

[15] Here we slightly abuse notion: the function $i(\xi)$ on the left hand side of the equation refers to the one used in equation (C.3) and the function $i(u)$ on the right hand side is the optimal policy function of investment rate.

Proof. *The above policy obviously satisfies the limited commitment constraint (6) after time $\tau(u_0)$. Below we prove that it also satisfies (6) before $\tau(u_0)$ and it satisfies the participation constraint, (5).*

To simplify notation, we use the expected utility representation of preferences. That is, use a monotonic transformation, $W = \frac{1}{1-\gamma}U^{1-\gamma}$ to measure utility. Under this convention, the outside option of the manager is $\frac{1}{1-\gamma}\bar{u}^{1-\gamma}K^{1-\gamma}$. Given the above policy, the highest utility that the manager can achieve at any time $s < \tau$ is:

$$\max_{\tau^*} E\left[\int_s^{\tau^*} e^{-r(t-s)} r \frac{1}{1-\gamma}(u_0K_0)^{1-\gamma}\,dt + e^{-r(\tau^*-s)}\frac{1}{1-\gamma}\bar{u}^{1-\gamma}K_{\tau^*}^{1-\gamma}\right],$$
$$\text{subject to}: \; dK_t = K_t\left[(\hat{\imath}-\delta)\,dt + \sigma dB_t\right]; \quad t \geq s$$

where the maximization is chosen over all B_t-adapted stopping times. Standard result implies that the value function of the above optimal stopping problem is of the form:

$$W(K) = \frac{1}{1-\gamma}(u_0K_0)^{1-\gamma} + \frac{1}{\alpha_1}\bar{u}^{1-\gamma}K^{*1-\gamma-\alpha_1}K^{\alpha_1},$$

and the optimal stopping rule is given by:

$$\tau^* = \inf\{t : K_t > K^*\},$$

where the optimal stopping threshold is

$$K^* = \left(\frac{\alpha_1}{\alpha_1 - 1 + \gamma}\right)^{\frac{1}{1-\gamma}} \frac{u_0}{\bar{u}}K_0.$$

Clearly, the optimal stopping time τ^* coincides with $\tau(u_0)$ in (C.6).

Note that the function $W(K)$ lies above $\frac{1}{1-\gamma}\bar{u}^{1-\gamma}K^{1-\gamma}$ for all $K < K^*$ and coincides with $\frac{1}{1-\gamma}\bar{u}^{1-\gamma}K^{1-\gamma}$ by the value matching condition. This shows that the limited commitment constraint (6) is satisfied at all times before $\tau(u_0)$. In addition, at time 0, $W(K_0) = \frac{1}{1-\gamma}(u_0K_0)^{1-\gamma} + \frac{1}{\alpha_1}\bar{u}^{1-\gamma}K^{*1-\gamma-\alpha_1}K_0^{\alpha_1} > \frac{1}{1-\gamma}(u_0K_0)^{1-\gamma}$ and therefore the participation constraint (5) is satisfied as well.

Finally, note that $\tau(u_0) \to \infty$ with probability one as $u_0 \to \infty$, as needed. ■

Proof of Lemma 2: To save notation, we consider only the case of limited commitment on the manager side. Other types of limited commitment can be proved by the same argument. Consider the HJB equation (27), the first order condition on g implies

$$g(u) = \frac{uv''(u)}{\gamma v'(u) + uv''(u)}.$$

Strict monotonicity and concavity of the value function implies $g(u) \in (0,1]$. On the boundary, $\lim_{u\to\bar{u}} g(u) = 1$ because $v''(u) \to -\infty$, as needed.

Proof of Proposition 3:

To see $i(u)$ is an increasing function of u, consider the HJB in (27). Note that the first order condition on investment implies that

$$h'(i(u)) = v(u) - uv'(u). \tag{C.11}$$

Taking derivatives with respect to u on both sides, we have:

$$i'(u) = \frac{-uv''(u)}{h''(i(u))} \geq 0$$

because $h''(i) > 0$ and $v''(u) \leq 0$. Note that $v''(u) \leq 0$ follows directly from the concavity of $V(U,K)$ in K.

To see that $\lim_{u\to\infty} i(u) = \hat{\imath}$ in the case with limited commitment on the manager side, note, that $v(u) \to \bar{v} - \frac{1}{r}u$ as $u \to \infty$. The first order condition, (C.11) then implies $i(u) \to \hat{\imath}$ as $u \to \infty$. The fact that $\lim_{u\to 0} i(u) = \hat{\imath}$ in the case with limited commitment on the shareholder side can be proved in a similar way.

Appendix D. Closed Form Solution

We first consider the case where Assumption 6 is not satified, that is,

$$\frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} \geq 1 + \frac{1}{r}\frac{\gamma}{\alpha_1 - 1}\left(\frac{\alpha_1+\gamma-1}{\alpha_1}\right)^{\frac{\gamma}{1-\gamma}}\bar{u}, \tag{D.1}$$

where $\alpha_1$is given by equation (D.9) in Appendix Appendix D. In this case, the solution of the optimal contract is given by the following lemma.

Lemma 8. 1. *For any $u \in [\bar{u}, \infty)$, the normalized compensation policy, $c(u)$, is defined by the unique solution of the following equation on $[c(\bar{u}), \infty)$:*

$$u^{1-\gamma} = c(u)^{1-\gamma} + \frac{1-\gamma}{\alpha_1+\gamma-1}c(\bar{u})^{\alpha_1}c(u)^{1-\gamma-\alpha_1}, \tag{D.2}$$

where $c(\bar{u}) = \left(\frac{\alpha_1+\gamma-1}{\alpha_1}\right)^{\frac{\gamma}{1-\gamma}}\bar{u}$ is the minimum compensation-to-capital ratio.

2. *Given $c(u)$, the normalized value function is given by*

$$v(u) = \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} - \frac{1}{r}c(u) + \frac{1}{r}\frac{1}{(1-\alpha_1)}c(\bar{u})^{\alpha_1}c(u)^{1-\alpha_1}. \tag{D.3}$$

3. *The optimal investment policy is given by*

$$i(u) = \hat{\imath} \quad for \quad all \quad u \in [\bar{u}, \infty). \tag{D.4}$$

4. *The Tobin's (average) Q is given by*

$$q_A(u) = \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} \quad for \quad all \quad u \in [\bar{u}, \infty). \tag{D.5}$$

Proof. See Appendix Appendix D. ■

The above proposition can be proved by a guess-and-verify approach. We conjecture that all firms invest at the maximum level $\hat{\imath}$, and construct the value function under this conjectured policy. We then verify that the value function satisfies the HJB equation and associated optimality conditions. Assume that the optimal investment is constant at $\hat{\imath}$. Define $\xi_t = \ln c(u_t)$. Then under the optimal contract,

$E\left[\int_0^\infty e^{-rt}\left(zK_t - I_t\right)dt\right] = \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}}$. Denote

$$G_C(\xi)K = E\left[\int_0^\infty e^{-rt}C_t dt \middle| \xi_0 = \xi, K_0 = K\right], \tag{D.6}$$

then by Lemma 6, $G_C(\xi)$ satisfies:

$$0 = e^\xi + [i-\delta-r]G_C(\xi) - \left(i - \delta + \frac{1}{2}\sigma^2\right)G_C'(\xi) + \frac{1}{2}\sigma^2 G_C''(\xi), \tag{D.7}$$

with $i = \hat{\imath}$. It is straight forward to show that the general solution to (D.7) is of the form

$$G_C(\xi) = \frac{1}{r}e^\xi + A_1 e^{(1-\alpha_1)\xi} + A_2 e^{(1-\alpha_2)\xi},$$

where α_1 and α_2 denote the solutions to the following quadratic equation

$$\frac{1}{2}\sigma^2\alpha + \left(i - \delta - \frac{1}{2}\sigma^2\right)\alpha - r = 0 \tag{D.8}$$

with $i = \hat{\imath}$, which are given by

$$\alpha_1 = \sqrt{\left(\frac{\hat{\imath}-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} - \left(\frac{\hat{\imath}-\delta}{\sigma^2} - \frac{1}{2}\right) > 1, \tag{D.9}$$

and

$$\alpha_2 = \sqrt{\left(\frac{\hat{\imath}-\delta}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} + \left(\frac{\hat{\imath}-\delta}{\sigma^2} - \frac{1}{2}\right) > 0. \tag{D.10}$$

The boundary conditions $G_C'\left(\bar{\xi}\right) = 0$, where $\bar{\xi} = \ln c(\bar{u})$ and $G_C(\xi) \to \frac{1}{r}e^\xi$ as $\xi \to \infty$ pins down the following unique solution:

$$G_C(\xi) = \frac{1}{r}e^\xi + \frac{1}{r}\frac{1}{\alpha_1 - 1}c(\bar{u})^{\alpha_1}e^{(1-\alpha_1)\xi}. \tag{D.11}$$

Therefore, if we define $G_V(\xi) = v\left(c^{-1}\left(e^\xi\right)\right)$, then

$$G_V(\xi) = \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} - \frac{1}{r}e^\xi - \frac{1}{r}\frac{1}{\alpha_1-1}c(\bar{u})^{\alpha_1}e^{(1-\alpha_1)\xi}. \tag{D.12}$$

The continuation utility of the manager is also a function of ξ. Define

$$W(\xi, K) = E_t\left[\int_t^\infty e^{-r(s-t)}rC_s^{1-\gamma}ds \middle| \xi_t = \xi, K_t = K\right], \tag{D.13}$$

then homogeneity implies

$$W(\xi, K) = G_U(\xi)K^{1-\gamma}, \tag{D.14}$$

and G_U satisfies (C.5) with $\rho = 1-\gamma$, $\eta(\xi) = e^{(1-\gamma)\xi}$ and $i(\xi) = \hat{\imath}$. The boundary conditions are: $G_U'(\xi) = 0$, where $\bar{\xi} = \ln c(\bar{u})$ and $G_C(\xi) \to e^{(1-\gamma)\xi}$ as $\xi \to \infty$. This gives:

$$G_U(\xi) = e^{(1-\gamma)\xi} + \frac{1-\gamma}{\alpha_1+\gamma-1}c(\bar{u})^{\alpha_1}e^{(1-\gamma-\alpha_1)\xi}. \tag{D.15}$$

Together, (D.12) and (D.15) imply the results in Proposition 4. One can show that

$$V_K(U, K) = G_V(\xi(u)) + \frac{1}{r}e^{\gamma\xi(u)}G_U(\xi(u)) \geq 1$$

for all $u \geq \bar{u}$. This verifies that $i(u) = \hat{\imath}$ is indeed optimal.

Proof of Proposition 4:

Now assume that assumption 6 is true. Suppose there exists u^* such that $V_K(U, K) = v(u) - uv'(u) < 1$ for $u < u^*$ and $v(u) - uv'(u) \geq 1$ for $u \geq u^*$. We denote $\xi^* = \ln c(u^*)$ and $\bar{\xi} = \ln c(\bar{u})$. Also, define $G_C(\xi)$ as in (D.6).

For $\bar{\xi} \leq \xi < \xi^*$, $G_C(\xi)$ has to satisfy (D.7) with $i = 0$ and the boundary condition $G_C'(\bar{\xi}) = 0$. The solution is of the form:

$$G_C(\xi) = \frac{1}{r}e^{\xi} - C_1 e^{(1-\beta_1)\xi} - C_2 e^{(1+\beta_2)\xi},$$

where β_1 and β_2 are solutions to (D.8) with $i = 0$:

$$\beta_1 = \sqrt{\left(\frac{\delta}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} + \left(\frac{\delta}{\sigma^2} + \frac{1}{2}\right) > 1,$$

$$\beta_2 = \sqrt{\left(\frac{\delta}{\sigma^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} - \left(\frac{\delta}{\sigma^2} + \frac{1}{2}\right) > 0.$$

For $\xi \geq \xi^*$, $G_C(\xi)$ has to satisfy (D.7) with with $i = \hat{\imath}$. Therefore, on (ξ^*, ∞), $G_C(\xi)$ is of the form:

$$G_C(\xi) = \frac{1}{r}e^{\xi} - D_1 e^{(1-\alpha_1)\xi} - D_2 e^{(1+\alpha_2)\xi}.$$

Using the boundary condition, $G_C(\xi) \to \frac{1}{r}e^{\xi}$ as $\xi \to \infty$, $D_2 = 0$ because $1 + \alpha_2 > 0$. Therefore, if we define $G_V(\xi) = v\left(c^{-1}\left(e^{\xi}\right)\right)$, then on $\left[\bar{\xi}, \xi^*\right)$,

$$G_V(\xi) = \frac{z}{r+\delta} - \frac{1}{r}e^{\xi} + C_1 e^{(1-\beta_1)\xi} + C_2 e^{(1+\beta_2)\xi},$$

and on $[\xi^*, \infty)$,

$$G_V(\xi) = \frac{z - \hat{\imath}}{r + \delta - \hat{\imath}} - \frac{1}{r}e^{\xi} + D_1 e^{(1-\alpha_1)\xi},$$

with the boundary condition

$$G_V'\left(\bar{\xi}\right) = 0. \tag{D.16}$$

In addition, $G_V(\xi)$ has to satisfy value matching and smooth pasting at ξ^*:

$$\lim_{\xi \to \xi^{*-}} G_V(\xi) = \lim_{\xi \to \xi^{*+}} G_V(\xi), \quad \lim_{\xi \to \xi^{*-}} G_V'(\xi) = \lim_{\xi \to \xi^{*+}} G_V'(\xi). \tag{D.17}$$

Similarly, define $G_U(\xi)$ as in (D.13) and (D.14), then on $\left[\bar{\xi}, \xi^*\right)$,

$$G_U(\xi) = e^{(1-\gamma)\xi} + A_1 e^{(1-\gamma-\beta_1)\xi} + A_2 e^{(1-\gamma+\beta_2)\xi},$$

and on $[\xi^*, \infty)$, [16]

$$G_U(\xi) = e^{(1-\gamma)\xi} + B_1 e^{(1-\gamma-\alpha_1)\xi}$$

with the boundary condition

$$G'_U(\bar{\xi}) = 0. \tag{D.18}$$

In addition, $G_V(\xi)$ has to satisfy value matching and smooth pasting at ξ^*:

$$\lim_{\xi \to \xi^{*-}} G_V(\xi) = \lim_{\xi \to \xi^{*+}} G_V(\xi), \quad \lim_{\xi \to \xi^{*-}} G'_V(\xi) = \lim_{\xi \to \xi^{*+}} G'_V(\xi). \tag{D.19}$$

Finally, at $\frac{U}{K} = \bar{u}$,

$$V_K(U,K) = G_V(\xi^*) + \frac{1}{r} e^{\gamma \xi(u)} G_U(\xi^*) = 1 \tag{D.20}$$

We have seven unknowns, A_1, A_2, B_1, C_1, C_2, D_1 and ξ^* to be determined by the seven equations, (D.16), (D.17), (D.18), (D.19) and (D.20). One can verify that under condition (31), $\xi^* > \bar{\xi} = \ln c(\bar{u})$, that is, $u^* > \bar{u}$. This completes the proof.

Note that

$$\begin{aligned} V_K(U,K) &= G_V(\xi(u)) + \frac{1}{r} e^{\gamma \xi(u)} G_U(\xi(u)) \\ &= \begin{cases} \frac{z}{r+\delta} + \left(C_1 + \frac{1}{r}A_1\right) c(u)^{1-\beta_1} + \left(C_2 + \frac{1}{r}A_2\right) c(u)^{1+\beta_2} & u \in [\bar{u}, u^*) \\ \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} + \left(D_1 + \frac{1}{r}B_1\right) c(u)^{1-\alpha_1} & u \in [u^*, \infty) \end{cases}. \end{aligned}$$

Therefore, if we define $\chi(u) = \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} - V_K(U,K)$ to be the marginal agency cost of investment, then

$$\chi(u) = \begin{cases} \frac{z-\hat{\imath}}{r+\delta-\hat{\imath}} - \frac{z}{r+\delta} - \left(C_1 + \frac{1}{r}A_1\right) c(u)^{1-\beta_1} - \left(C_2 + \frac{1}{r}A_2\right) c(u)^{1+\beta_2} & u \in [\bar{u}, u^*) \\ -\left(D_1 + \frac{1}{r}B_1\right) c(u)^{1-\alpha_1} & u \in [u^*, \infty) \end{cases} \tag{D.21}$$

Using the boundary conditions we can verify $u^* = \bar{u}$ if and only if

$$\chi(\bar{u}) = \frac{1}{r} \frac{\gamma}{\alpha_1 - 1} \left(\frac{\alpha_1 + \gamma - 1}{\alpha_1}\right)^{\frac{\gamma}{1-\gamma}} \bar{u} = 1$$

Proof of Proposition 5: The maximum marginal agency cost of investment is given by:

$$\chi\left(\bar{u}; r, \hat{\imath}, \sigma^2\right) = \frac{1}{r} \frac{\gamma}{\alpha_1 - 1} \left(\frac{\alpha_1 + \gamma - 1}{\alpha_1}\right)^{\frac{\gamma}{1-\gamma}} \bar{u}$$

Under Assumptions 4 and 5, $\alpha_1 > 1$ and $\alpha_1 + \gamma - 1 > 0$. Therefore, $\frac{\partial \chi}{\partial \bar{u}} = \frac{1}{r} \frac{\gamma}{\alpha_1 - 1} \left(\frac{\alpha_1+\gamma-1}{\alpha_1}\right)^{\frac{\gamma}{1-\gamma}} > 0$.

[16] Here we impose the boundary condition $\lim_{\xi \to \infty} G_V(\xi) = e^{(1-\gamma)\xi}$, and use the fact $1 - \gamma + \alpha_2 > 0$ to eliminate the additional term in the general solution.

We prove the rest of the lemma by showing $\frac{\partial \ln \chi}{\partial r} < 0$, $\frac{\partial \ln \chi}{\partial \hat{\imath}} > 0$ and $\frac{\partial \ln \chi}{\partial \sigma^2} > 0$. Note first,

$$\ln \chi = -\ln r + \ln \gamma - \ln (\alpha_1 - 1) + \frac{\gamma}{1-\gamma} \ln \left(\frac{\alpha_1 + \gamma - 1}{\alpha_1} \right) + \ln \bar{u}.$$

Therefore,

$$\begin{aligned} \frac{\partial \ln \chi}{\partial \alpha_1} &= -\frac{1}{\alpha_1 - 1} + \frac{\gamma}{1-\gamma} \left[\frac{1}{\alpha_1 + \gamma - 1} - \frac{1}{\alpha_1} \right] \\ &= -\frac{1}{\alpha_1 - 1} + \frac{\gamma}{(\alpha_1 + \gamma - 1)\,\alpha_1} \\ &= -\frac{\gamma + \alpha_1 (\alpha_1 - 1)}{\alpha_1 (\alpha_1 - 1)(\alpha_1 + \gamma - 1)} < 0. \end{aligned}$$

Using the fact that α_1 is the solution to the quadratic equation (D.8), we have:

$$\frac{\partial \alpha_1}{\partial r} = \frac{1}{(\hat{\imath} - \delta) + \left(\alpha_1 - \frac{1}{2}\right)\sigma^2} > 0,$$

$$\frac{\partial \alpha_1}{\partial \hat{\imath}} = -\frac{\alpha_1}{(\hat{\imath} - \delta) + \left(\alpha_1 - \frac{1}{2}\right)\sigma^2} < 0,$$

and

$$\frac{\partial \alpha_1}{\partial \sigma^2} = -\frac{\frac{1}{2}\alpha_1 (\alpha_1 - 1)}{(\hat{\imath} - \delta) + \left(\alpha_1 - \frac{1}{2}\right)\sigma^2} < 0.$$

Therefore,

$$\frac{\partial \ln \chi}{\partial r} = -\frac{1}{r} + \frac{\partial \ln \chi}{\partial \alpha_1} \frac{\partial \alpha_1}{\partial r} < 0,$$

$$\frac{\partial \ln \chi}{\partial \hat{\imath}} = \frac{\partial \ln \chi}{\partial \alpha_1} \frac{\partial \alpha_1}{\partial \hat{\imath}} > 0,$$

and

$$\frac{\partial \ln \chi}{\partial \sigma^2} = \frac{\partial \ln \chi}{\partial \alpha_1} \frac{\partial \alpha_1}{\partial \sigma^2} > 0,$$

as needed.

Fig. D.1. Value Functions: The Case of Limited Commitment

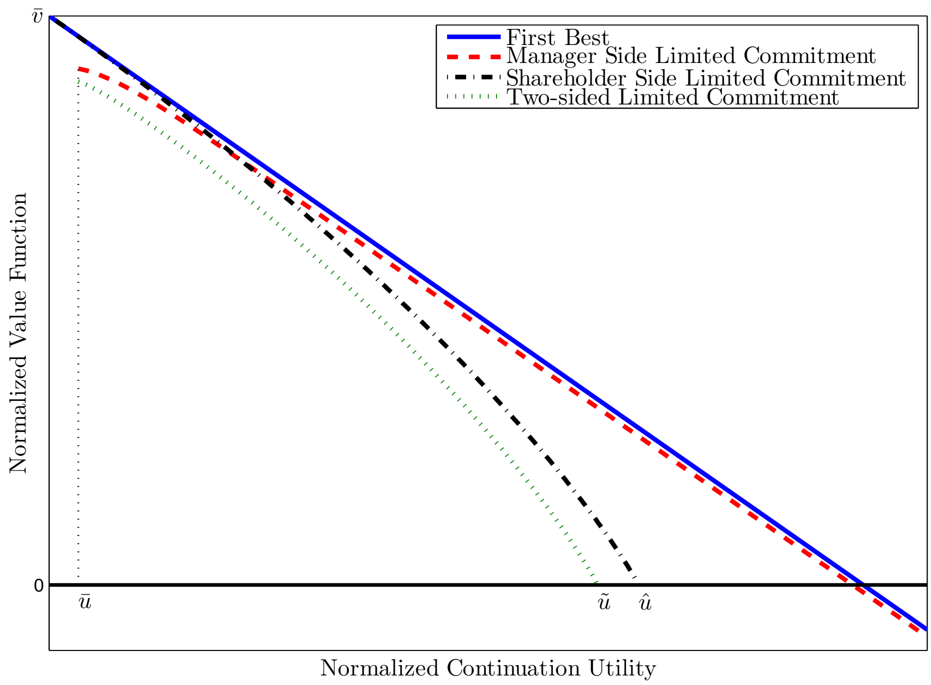

Figure D.1 plots the normalized value functions for the first best, manager side limited commitment, shareholder side limited commitment, and two-sided limited commitment case, respectively. Figures D.1-D.2 are based on a quadratic adjustment cost function: $h(i) = i + \phi i^2$, where $\phi = 5$. Other parameters are: $\gamma = 2$, $\beta = 0.08$, $z = 0.24$, $\theta = 1$, $\sigma = 36\%$, and $\delta = 9\%$.

Fig. D.2. Dynamics of Normalized Continuation Utility: Limited Commitment

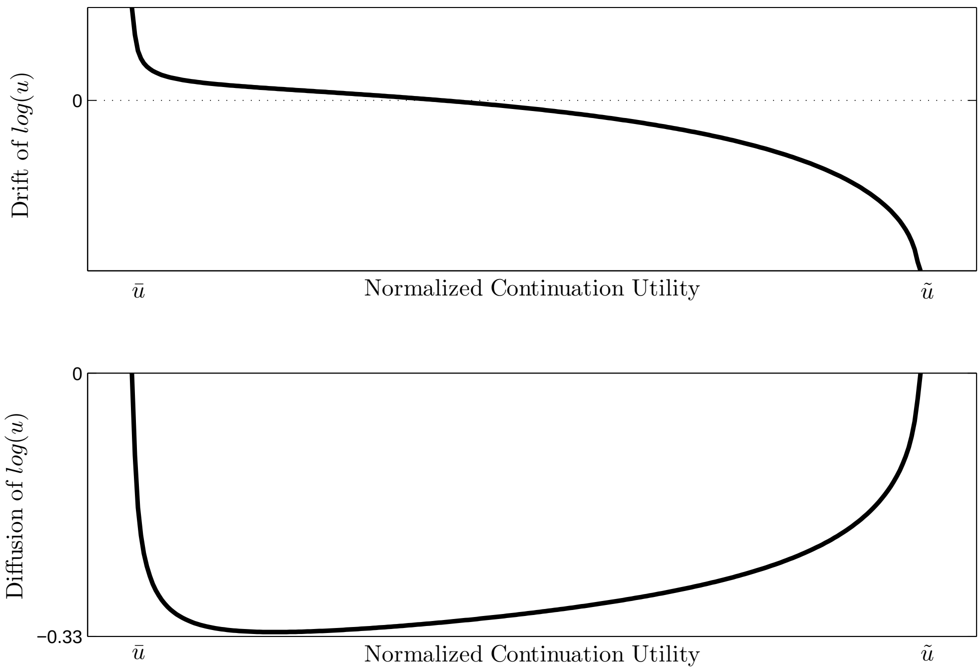

Figure D.2 illustrates the dynamics of the normalized continuation utility, u, for the case of two-sided limited commitment. The drift of u is strictly positive on the left, monotonically decreasing, and strictly negative on the right. The diffusion is zero on the boundaries and strictly negative in the interior.

Fig. D.3. Investment Policy: First Best and One-Sided Limited Commitment

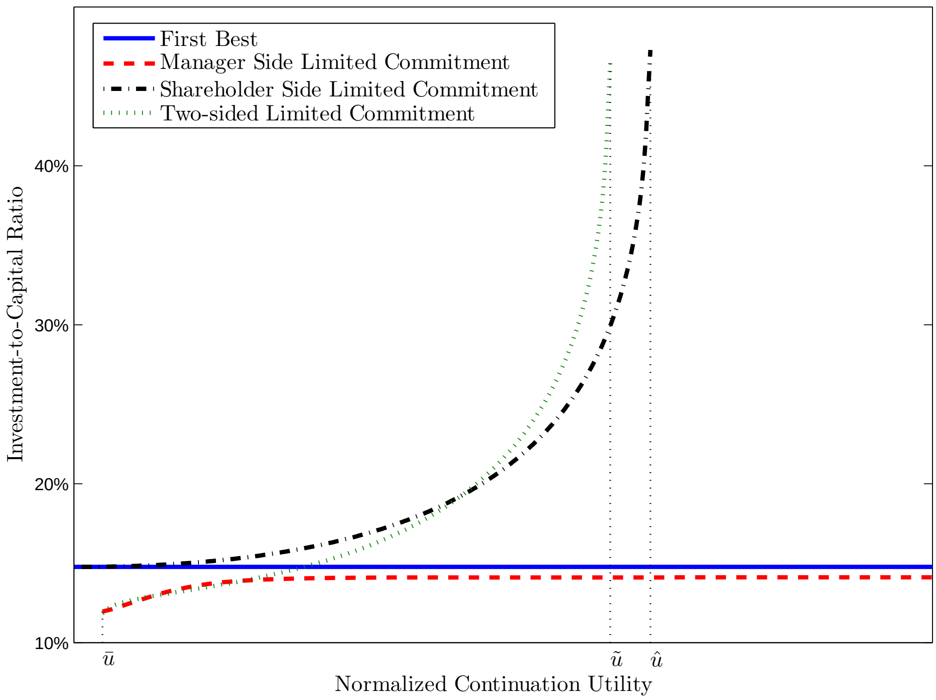

Figure D.3 plots the optimal investment rate for the first best, manager side limited commitment, shareholder side limited commitment, and two-sided limited commitment case, respectively.

Fig. D.4. Marginal Agency Cost of Investment and Optimal Investment Policy

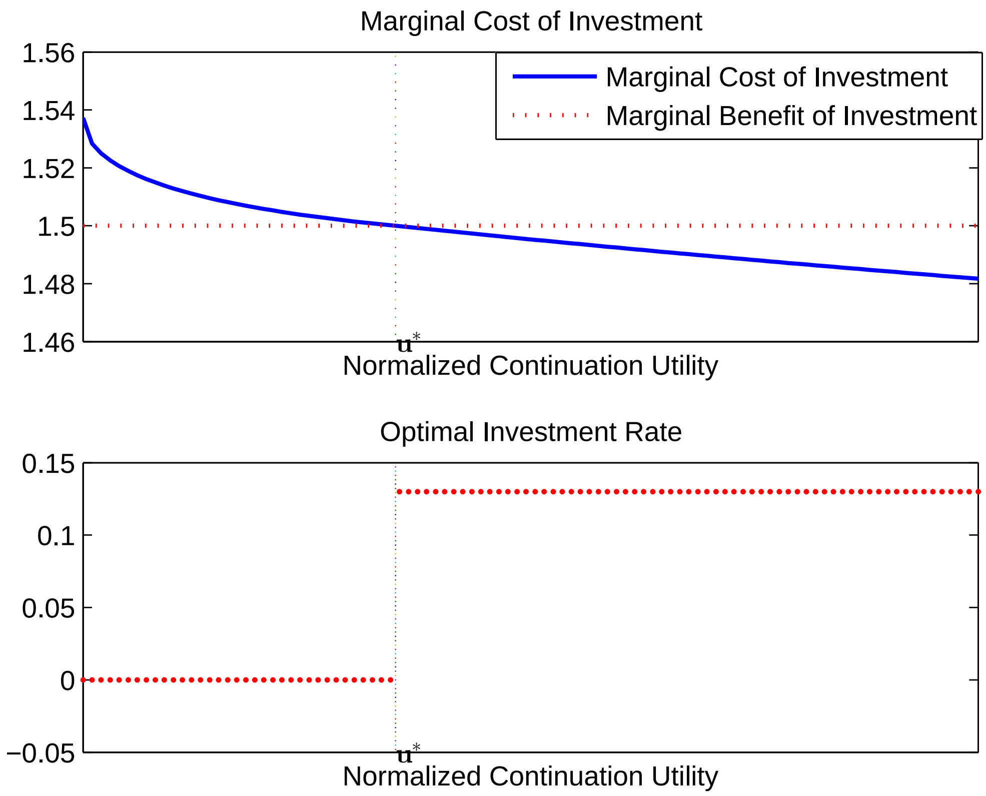

Figure D.4 plots the marginal cost of investment (top panel) and the optimal investment rate (bottom panel) in the example with a closed-form solution. We use the following parameters: $\gamma = 0.5$, $\beta = 0.04$, $z = 0.145$, $\theta = 1$, $\sigma = 25\%$, $\hat{\imath} = 0.13$, and $\delta = 10\%$ for Figures D.4-D.5.

Fig. D.5. Tobin's Q and Firm Size

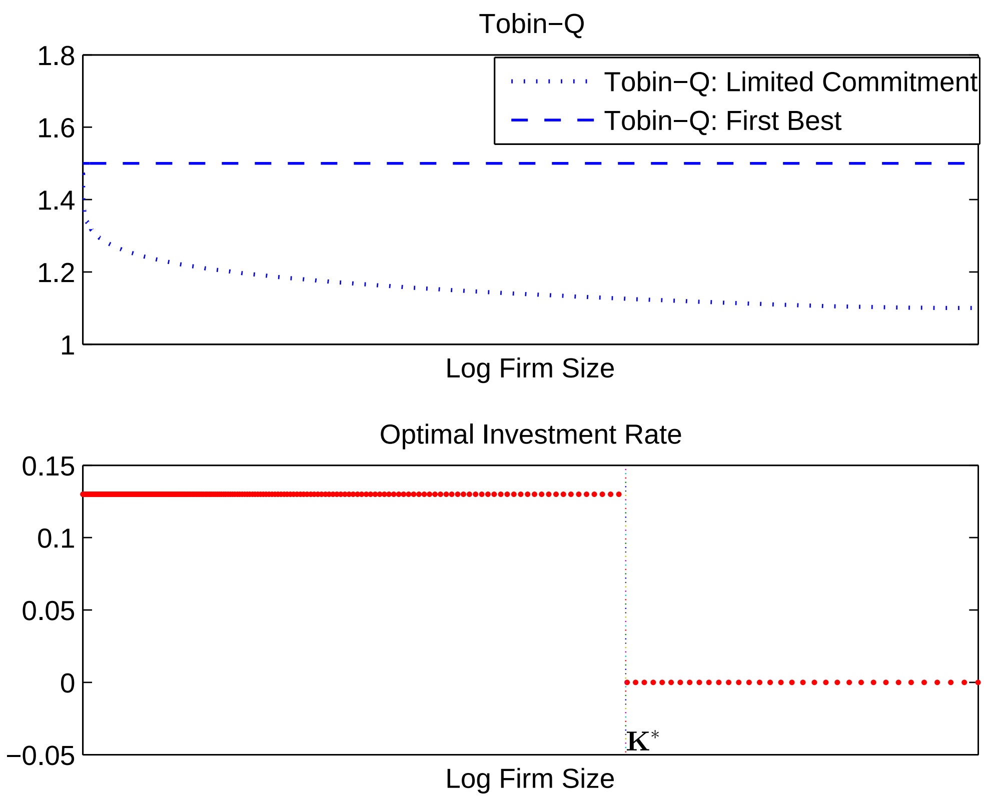

Figure D.5 plots the Tobin's Q as a function of firm size (top panel) and optimal investment rate as a function of firm size (bottom panel). They are based on the example with a closed-form solution in Section 4.

Fig. D.6. Power Law of Firm Size

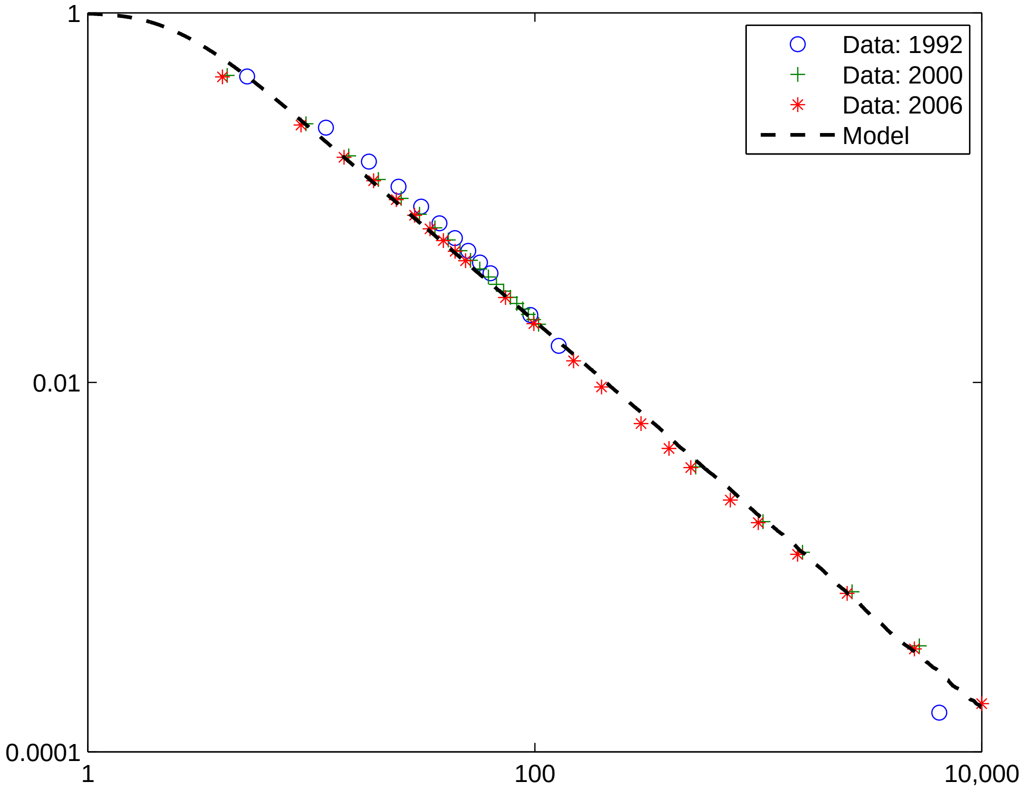

Figure D.6 plots the power law of the distribution of firm size in the data and in the model. Firm size is measured by the total number of employees reported by the Small Business Administration in the year 1992 (circles), 2000 (plus signs), and 2006 (stars). The power law of firm size in the model is represented by the dashed line.

Fig. D.7. Firm Growth and Firm Size

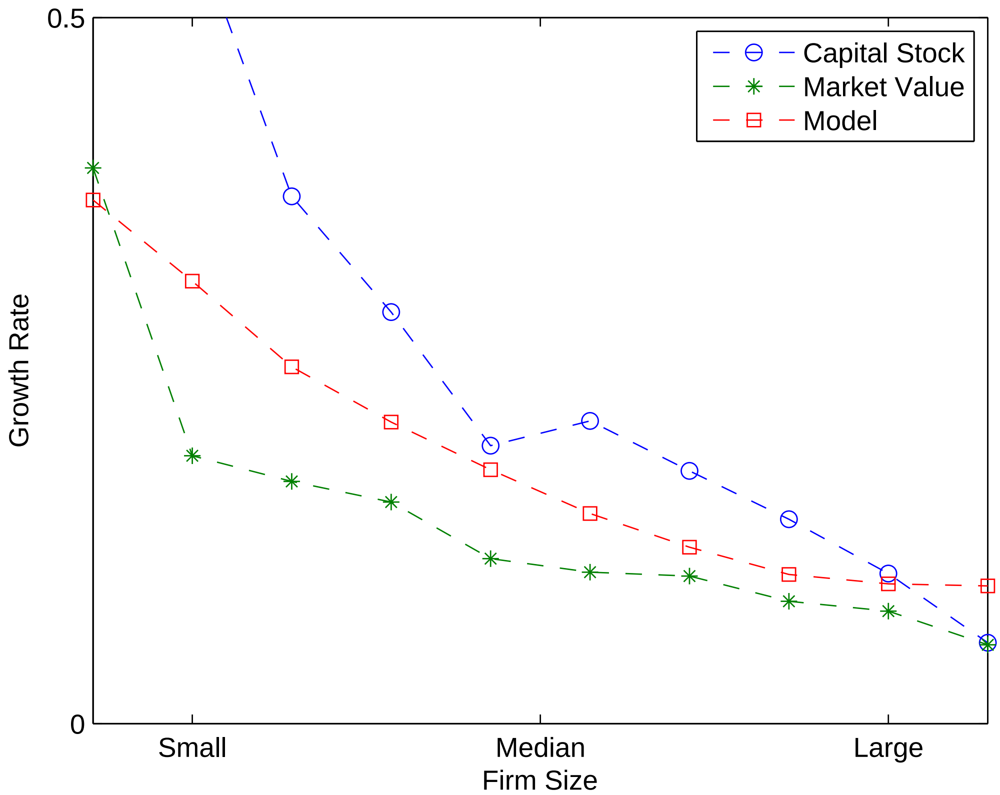

Figure D.7 plots firms' average growth rate as a function of their sizes. In the data, we use all firms listed in the COMPUSTAT data base during the period 1950-2009. We sort firms into ten size deciles for each year, and compute the average growth rate for all firms in each decile and average across all years. Firm size is measured by total capital stock (circles) and the total value of the firm's asset (stars), respectively. Capital stock is computed as the total asset (at) minus the current asset (act) of the firm. The total value of firm's asset is constructed as the sum of the firm's market value of equity and the book value of debt. The squares represent the growth rate-size relationship generated by our model.

Fig. D.8. Tobin's Q and Firm Size

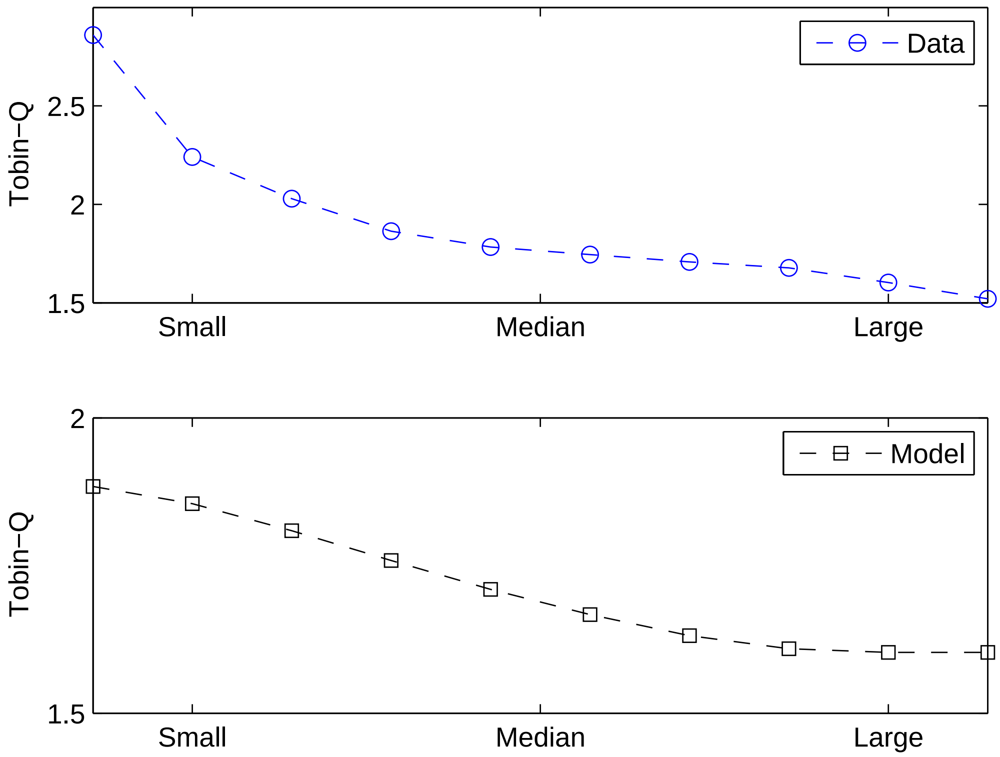

Figure D.8 plots the relationship between Tobin's Q and firm size in the data (top panel) and in the model (bottom panel). We use all firms listed in the COMPUSTAT data base during the period 1950-2009. Firm size is measured by the total capital stock calculated as total asset minus current asset. Our construction of Tobin's Q follows Erickson and Whited (2000).

Fig. D.9. Marginal Q as a Function of Average Q

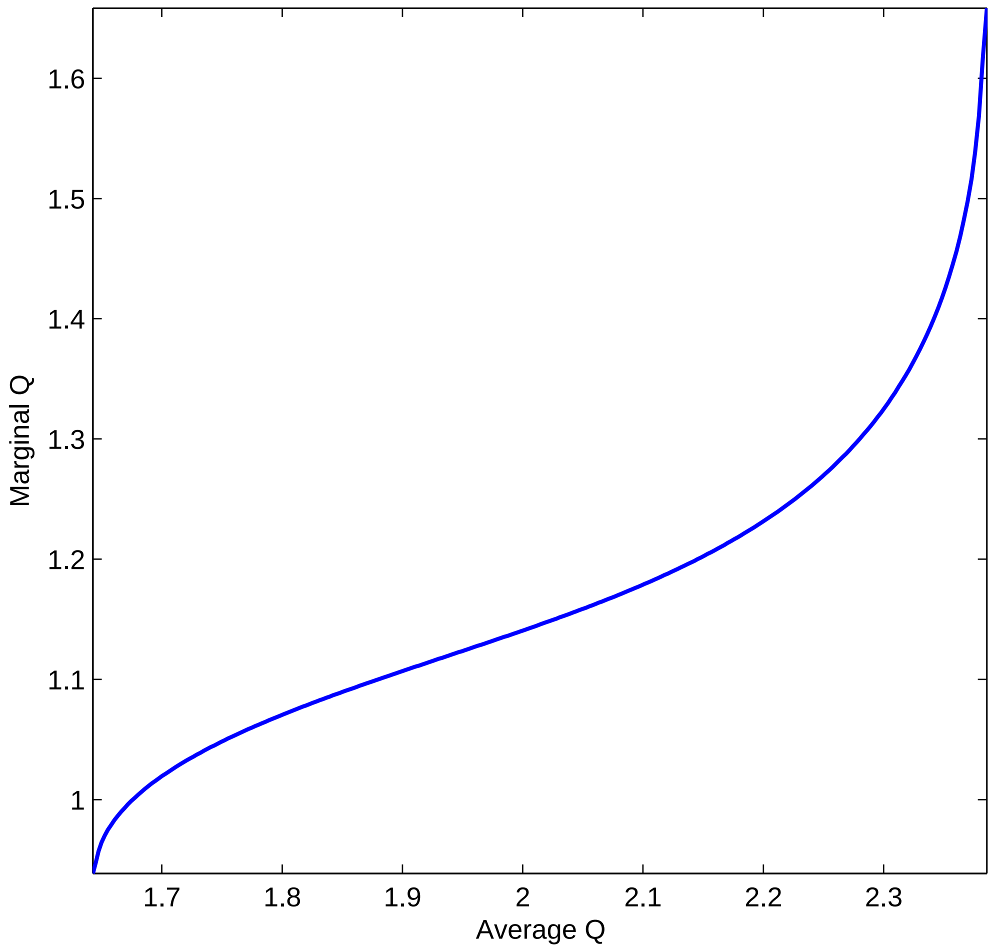

Figure D.9 plots the marginal Q as a function of average Q implied by our model.

Table D.1

Investment Regressions

	Tobin's Q	CF/K	R^2
Data	0.014	0.006	0.038
	(52.28)	(51.53)	
Model	0.60	1.08	0.801
	(2383)	(2032)	

Table 1 documents the results of the investment regression (36). Investment is measured by total capital expenditure (capxv) minus sales of property, plant and equipment (sppe). Cash flow is measured by total sales (sales) minus cost of good sold (cogs). Tobin's Q is constructed in the same way as in Erickson and Whited (2000). Our regression include all 147847 firm-year observations in the COMPUSTAT database during the period 1950-2009 for which all of the above variables are available. T-statistics of the estimated regression coefficients are listed in parentheses. We use the COMPUSTAT data set.

Table D.2

History Dependence of CEO Compensation

	Current Size	Max Size	Min Size	R^2
Data	-0.009	0.038	0.032	0.129
	(-2.95)	(12.50)	(3.67)	
Model	-0.005	0.032	0.034	0.969
	(-66.10)	(480)	(48)	

Table 2 documents regression of CEO compensation on current and historical size of the firm, (37). Here, CEO compensation is measured by the total compensation including option grants (TDC1) of the top three executives of the firm reported in EXECUCOMP data base. Firm size is measured by the sum of the total market value of equity and the total book value of debt of the firm. The historical best and worst performances of the firm are measured the highest and lowest historical values of the firm in the last ten years, respectively. T-statistics of the estimated regression coefficients are listed in parentheses.

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