A tractable model of limited enforcement and the life-cycle of firms

Abstract

We develop a continuous-time model of optimal lending contracts under limited enforcement and provide closed-form solutions. We characterize the dynamics of firms' growth rate, Tobin's Q, and capital structure over their life cycle.

1 Introduction

This paper studies how limited enforcement of lending contracts affects the life cycle and long-run behaviors of firms. We build a continuous-time dynamic model in which an entrepreneur borrows from an outside investor to finance capital and to invest in productive projects. We assume that the lending contract can only be partially enforced. In particular, the entrepreneur can default on loan contracts and abscond with at least part of the capital input funds without legal recourse. We characterize in closed form the optimal dynamic contract subject to limited enforcement. We show that under the optimal contract, firms' life cycle consists of two stages. The first stage of firms' life cycle is characterized by high growth rates, a high Tobin's Q, low dividend payments, and low leverage. Entrepreneurs are financially constrained, and their firms operate at suboptimal levels in the first stage. They do not draw any dividend payment from the firm and save as much as possible to relax the borrowing constraint. In this stage, firm growth is driven by both productivity growth and convergence to the efficient level of operation. In the second stage, the firm reaches maturity and its efficient level of operation. In this stage, entrepreneurs have accumulated enough net worth and reached their maximum borrowing capacity. Firm growth in this stage is only driven by productivity growth and not by improvements in the scale of operation. As a result, firms in the mature stage have low growth rates, a low Tobin's Q, and high leverage. These predictions of our model are consistent with the empirical findings in the literature. The model also provides several testable implications on how the time to reach maturity and the long-run behaviors of firms depend on the primitive technology and contracting parameters.

The paper most closely related to ours is by Albuquerque and Hopenhayn [2004], who also study a firm dynamics model with limited enforcement. Our paper is different from theirs because it provides an analytical solution to the firm dynamics, which allows additional implications, including how the capital structure evolves over the firm's life cycle and how it depends on the production technology or the severity of the financial market friction in the long run.

2 The model

Time is continuous, infinite, and denoted $t \in [0, \infty)$. An entrepreneur operates a firm that uses a Cobb-Douglas technology to produce output from capital. Given a sequence of capital inputs $\{K_t\}$, the operating profit of his firm at time t, denoted by $\pi(Z_t, K_t)$, is

$$\pi(Z_t, K_t) = Z_t^{1-\alpha} K_t^{\alpha} - (r + \delta) K_t.$$

Here, Z_t is the firm's productivity, $\alpha \in (0, 1)$ is the returns to scale parameter, $r > 0$ is the interest rate, $\delta > 0$ is the capital depreciation rate; and $r + \delta$ is the user's cost of a unit of capital. We assume that Z_t follows a geometric Brownian motion with constant expected growth rate μ and volatility σ:

$$dZ_t = Z_t [\mu dt + \sigma dB_t]. \tag{1}$$

We assume that the entrepreneur does not have any initial wealth and has to borrow from an outside investor. A lending contract specifies the capital input, dividend payment to the entrepreneur, and loan repayment to the outside investor as functions of the history of the realizations of shocks. We use C_t to denote the cumulative dividend payment to the entrepreneur. Limited liability implies that C_t must be non-decreasing. Given $\{K_t\}_{t=0}^{\infty}$ and $\{C_t\}_{t=0}^{\infty}$, the resource constraint implies that the loan repayment to the investor at time t must be $\pi(Z_t, K_t) dt - dC_t$; therefore, we can without loss of generality write a loan contract as $\{K_t, C_t\}_{t=0}^{\infty}$.

Both the entrepreneur and investor are risk-neutral. Under the contract, the entrepreneur's expected utility is

$$E_0 \left[\int_0^{\infty} e^{-\beta t} dC_t \right],$$

with $\beta > 0$ being the discount rate and E_0 being the time zero expectation operator. The expected payoff of the investor is

$$E_0 \left[\int_0^{\infty} e^{-rt} \left[\left(Z_t^{1-\alpha} K_t^{\alpha} - (r + \delta) K_t \right) dt - dC_t \right] \right]. \tag{2}$$

As is standard in the optimal contracting literature, we assume $r < \beta$ so that the investor is more patient.

To capture the idea that the lending contract cannot be fully enforced, we assume that the entrepreneur can always default by absconding with a fraction of the capital, θK_t, without any legal recourse. Here, $\theta \in [0,1]$ is a parameter of the enforceability of contracts, and $\theta = 0$ corresponds to the case of perfect enforcement. In addition, upon default, the project evaporates; therefore, default is always socially inefficient.

Assumption 1. *$\beta > r > \mu > 0$ and $\theta < \frac{r-\mu}{\beta-\mu}$.*

The first inequality guarantees a finite payoff to the investor; the second is for technical reasons and is satisfied if θ is small or β and r are close. The entrepreneur's payoff upon default is proportional to θ, which indicates how severe the limited enforcement problem is.

3 The optimal contract

3.1 State variables and normalization

By following the literature, we define the entrepreneur's continuation utility:

$$U_t = E_t\left[\int_t^\infty e^{-\beta(s-t)} dC_s\right] \text{ for all } t \geq 0,$$

and the martingale representation theorem implies[1]

$$dU_t = \beta U_t dt - dC_t + g_t U_t \sigma dB_t \tag{3}$$

with $\{g_t\}$ being a predictable process which indicates the sensitivity of U_t to the productivity shocks. The borrowing constraint preventing default is

$$U_t \geq \theta K_t \text{ for all } t \geq 0, \tag{4}$$

which implies an upper bound restricting the capital input to be inefficiently low. So, the investor designs the optimal contract that maximizes her expected payoff, (2). Under the contract, U_t evolves according to (3), and $Z_0 = 1$ and $U_0 = \underline{U}$ with $\underline{U}$ being the initial expected utility promised to the entrepreneur. Furthermore, the borrowing constraint (4) is satisfied. Let $V(Z,U)$ be the value function of the investor. The linear preferences and concave production function imply the following result.[2]

Lemma 1. *$V(Z,U)$ is concave in U.*

The maximization problem is homogeneous in Z so that

$$V(Z,U) = Zv\left(\frac{U}{Z}\right)$$

[1]See DeMarzo and Sannikov [2006] and Sannikov [2008].

[2]The proofs of some of the results are straightforward and omitted, but are available via email upon request.

with $v(u)$ being the normalized value function. Here, $u = \frac{U}{Z}$ is the normalized continuation utility of the entrepreneur, the ratio of his future payments to the scale of the firm, which is interpreted as his stake in the firm. Likewise, we define the normalized policies $k = \frac{K}{Z}$, $dc = \frac{dC}{Z}$, and $\underline{u} = \frac{U}{Z_0} = \underline{U}$. Thus, (3) implies

$$du_t = u_t \left[\left((\beta - \mu) + (g_t - 1)\, \sigma^2 \right) dt + (g_t - 1)\, \sigma dB_t \right] - dc_t, \tag{5}$$

and the borrowing constraint (4) implies

$$k_t \leq \frac{u_t}{\theta} \text{ for all } t \geq 0. \tag{6}$$

According to the investor's contract design problem, $v(u)$ satisfies the following HJB differential equation:

$$\begin{aligned} 0 = \max_{k \in \left[0, \frac{u}{\theta}\right], g, dc \geq 0} \quad & k^{\alpha} - (r + \delta)\, k - (r - \mu)\, v\,(u) + (\beta - \mu)\, uv'\,(u) \\ & + \frac{1}{2} u^2 v''(u)(g-1)^2 \sigma^2 - (1 + v'(u))\, dc. \end{aligned} \tag{7}$$

In the first-best case without limited enforcement, $k_t = k^*$ with

$$k^* = \arg\max_{k} k^{\alpha} - (r + \delta)\, k = \left(\frac{\alpha}{r + \delta} \right)^{\frac{1}{1-\alpha}},$$

and the rate of operating profit is $\pi^* Z_t$, with $\pi^* = (1 - \alpha) \left(\frac{\alpha}{r+\delta} \right)^{\frac{\alpha}{1-\alpha}}$. The initial utility of the entrepreneur is fully paid off at $t = 0$, so that the first-best normalized value function, $v^{FB}(u)$, satisfies

$$v^{FB}(u) = \frac{\pi^*}{r - \mu} - u \text{ for } u \geq 0.$$

Clearly, this contract violates the borrowing constraint, (6), as the entrepreneur's stake decreases to zero immediately.

3.2 The optimal contract with limited enforcement

We heuristically discuss the characterization of $v(u)$ and the optimal contract with limited enforcement, which are formally stated in Proposition 1. According to Lemma 1, $v''(u) \leq 0$. Therefore, equation (7) implies $g_t = 1$ and therefore neither u_t nor the normalized policies respond to the productivity shocks.

Since the investor can always pay off the payments promised to the entrepreneur immediately by a lump-sum transfer, $v(u) \geq v(u - dc) - dc$ for all $dc > 0$, and thus $v'(u) \geq -1$.[3] So, concavity of v implies that there exists a level $\hat{u} > 0$ such that $v'(u) > -1$

[3] See DeMarzo and Sannikov [2006] and He [2009] for similar arguments.

if and only if $u \in [0, \hat{u})$. Hence,

$$dc_t = 0 \text{ and } du_t = (\beta - \mu)\, dt \text{ if } u_t \in [0, \hat{u})\,. \tag{8}$$

Namely, u_t grows at a constant rate, $\beta - \mu$, until it reaches $\hat{u}$ where the contract starts paying the entrepreneur at a constant rate $dc_t = (\beta - \mu)\, \hat{u} dt$ so that u_t is constant at $\hat{u}$. Intuitively, deferring payments makes u_t grow and relaxes the borrowing constraint (6) when $u_t < u^* \equiv \theta k^*$, allowing more efficient levels of capital to be financed without default. However, doing so is costly, as the entrepreneur is less patient. Therefore, the contract starts paying dividends at level $\hat{u}$ where the marginal cost and benefit of deferring payments are balanced.[4] Over $[0, \hat{u}]$ the HJB equation (7) can be rewritten as

$$0 = \left(\frac{u}{\theta}\right)^{\alpha} - (r + \delta)\, \frac{u}{\theta} - (r - \mu)\, v\,(u) + (\beta - \mu)\, u v'\,(u) \tag{9}$$

with $v(0) = 0$ and $v'\,(\hat{u}) = -1$.

To determine the boundary $\hat{u}$, for any $u \geq 0$, let's consider a suboptimal contract under which $dc_t = (\beta - \mu)\, u dt$ with $u_t = u$ for all $t \geq 0$. Furthermore, the capital input is optimal given the compensation policy and the constraint (6) such that $k = \min\left\{\frac{u}{\theta}, k^*\right\}$. The investor's normalized value function under this type of contract, $\underline{v}\,(u)$, satisfies

$$\underline{v}(u) = \begin{cases} \frac{1}{r-\mu}\left[\left(\frac{u}{\theta}\right)^{\alpha} - (r+\delta)\,\frac{u}{\theta} - (\beta - \mu)\, u\right] & \text{if } u \in [0, u^*] \\ \frac{\pi^*}{r-\mu} - \frac{\beta-\mu}{r-\mu} u & \text{if } u \in [u^*, \infty)\,. \end{cases} \tag{10}$$

Notice that the optimal contract switches to this type when u_t reaches $\hat{u}$. Hence $v\,(\hat{u}) = \underline{v}\,(\hat{u})$ and the smooth pasting condition[5] implies $v'\,(\hat{u}) = \underline{v}'\,(\hat{u}) = -1$. Consequently,

$$\hat{u} = \theta \left[\frac{\alpha}{r + \delta + \theta\,(\beta - r)}\right]^{\frac{1}{1-\alpha}}. \tag{11}$$

Figure 1 illustrates the construction of $v(u)$. Now we have the following explicit characterization of v and the optimal contract.[6]

Proposition 1. *The contract promising the normalized utility $\underline{u} \in (0, \hat{u}]$ to the entrepreneur takes the following form: u_t evolves according to (8) until it is absorbed at $\hat{u}$; $dc_t = 0$ if $u_t < \hat{u}$ and $dc_t = (\beta - \mu)\hat{u} dt$ if $u_t = \hat{u}$; and $k_t = k\,(u_t) = \frac{u_t}{\theta}$. If $\underline{u} > \hat{u}$, an immediate transfer, $dc_0 = \underline{u} - \hat{u}$, is paid. For $u \in [0, \hat{u}]$, the investor's normalized value function $v\,(u)$ satisfies*

$$v(u) = \frac{r + \delta}{\beta - r}\left(\frac{u}{\theta}\right) + \frac{1}{r - \mu - \alpha(\beta - \mu)}\left(\frac{u}{\theta}\right)^{\alpha} + C^{v} u^{\frac{r-\mu}{\beta-\mu}}, \tag{12}$$

[4] Obviously, $\hat{u} < u^*$ because the marginal cost is strictly positive.

[5] See Dixit [1999] for the smooth pasting condition of the optimal stopping problem.

[6] Proposition 1 is the verification theorem of the value function. The proof is standard and omitted.

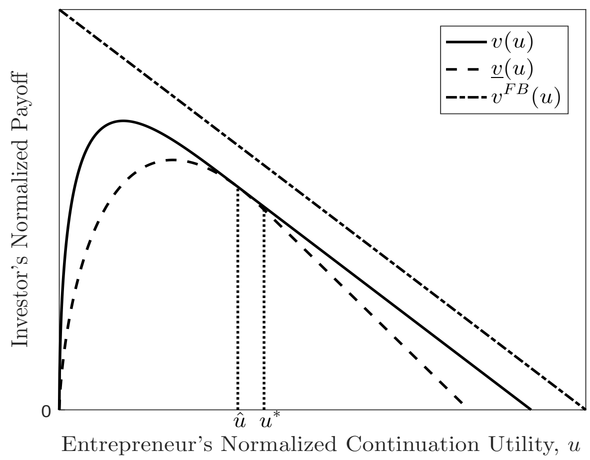

Figure 1: Characterization of the normalized value function, $v(u)$.

with constant C^v *being determined by*

$$v'(\hat{u}) = -1 \text{ or } v(\hat{u}) = \underline{v}(\hat{u}) \tag{13}$$

for $u \in [\hat{u}, \infty)$, $v'(u) = -1$, *and* $v(u) = v(\hat{u}) - (u - \hat{u})$.

We make an additional assumption about the entrepreneur's initial utility level, $\underline{u}$, which implies that the contract starts efficiently and is renegotiation proof.[7]

Assumption 2. $\underline{u} \in [\arg\max_{\tilde{u}} v(\tilde{u}), \hat{u}]$.

4 Model implications

In this section, we discuss the implications of the model for firm dynamics and capital structure.

4.1 The total firm value, Tobin's Q, and leverage

Define the time-t total firm value, $W(Z_t, U_t)$, to be the present value of the future operating profit at t:

$$W(Z_t, U_t) = \int_t^\infty e^{-r(s-t)} \left(Z_s^{1-\alpha} K_s^\alpha - (r+\delta) K_s \right) ds.$$

Let $w(u) = W(Z,U)/Z$ be the normalized total firm value. We then have the following characterization of $w(u)$.

[7]Notice that if $\underline{u} < \arg\max_{\tilde{u}} v(\tilde{u})$, it is mutually beneficial to raise the entrepreneur's initial utility.

Proposition 2. *Let* $\hat{k} = \frac{\hat{u}}{\theta}$ *and define* $\hat{\pi}$ *such that* $\hat{\pi} = \hat{k}^{\alpha} - (r+\delta)\,\hat{k}$*. Then, for* $u \in [0, \hat{u}]$

$$w(u) = \frac{r+\delta}{\beta - r}\left(\frac{u}{\theta}\right) + \frac{1}{r-\mu-\alpha(\beta-\mu)}\left(\frac{u}{\theta}\right)^{\alpha} + C^{w} u^{\frac{r-\mu}{\beta-\mu}}, \tag{14}$$

where the constant C^w *is determined by*

$$w(\hat{u}) = \frac{\hat{\pi}}{r-\mu} \tag{15}$$

$w'(u) > 0$ *for* $u \in [0, \hat{u})$.

The expression of $w(u)$ can be derived similarly to the way we derive $v(u)$, and $w(u)$ increases with u because a higher level of u relaxes the borrowing constraint, making production more efficient. Define the Tobin's Q as the value of one unit of capital:

$$q(u) = \frac{W(Z,U)}{K(Z,U)} = \frac{w(u)}{k(u)}.$$

Let $Y(Z_t, U_t)$ be the equity of the firm, the present value of the future transfer payments to the entrepreneur. The normalized equity is $y(u) = Y(Z,U)/Z$ and, clearly, $y(u) = w(u) - v(u)$.[8] We define the leverage rate in this model as

$$l(u) \equiv \frac{K(Z,U) - Y(Z,U)}{W(Z,U)} = \frac{k(u) - y(u)}{w(u)}.$$

Notice that K is the total capital financing and Y is the value of the equity. Therefore, $l(u)$ is the ratio of debt to the total asset value.

4.2 The age- and size-dependent firm behaviors

Let $\hat{T}$ be the age when u_t reaches $\hat{u}$. According to (5), $\hat{T} = \ln(\hat{u}/\underline{u})/(\beta - \mu)$. Therefore, we have the following age-dependent firm behaviors.

Proposition 3. *Under the optimal contract:*

(a) The growth rate of K_t *is* β *for* $t \in \left[0, \hat{T}\right)$ *and* μ *for* $t \in \left[\hat{T}, \infty\right)$*;*

(b) The Tobin's Q, $q(u_t)$*, decreases with* t*;*

(c) The leverage rate, $l(u_t)$*, increases with* t*.*

Proof. See Appendix A. □

Before $\hat{T}$, the entrepreneur does not receive any dividend, and his promised payments are deferred. So his stake in the firm goes up, enhancing the firm's creditworthiness and

[8] Notice that $y(u) \neq u$ because $\beta > r$.

allowing more efficient capital input. In this stage, the firm growth is fast and driven by relaxation of the borrowing constraint and productivity growth. Once $\hat{T}$ is reached, the capital financing reaches it maximum capacity and the contract starts paying dividends. The firm growth slows down and is only driven by productivity growth. We say that the firm is *mature* at $\hat{T}$. Proposition 3 show that mature firms have a lower Tobin's Q and are more leveraged. Since the age and size are positively correlated in our model, everything has the same dependence on size.

4.3 Time to reach maturity and long-run behaviors

The following proposition shows how the contracting parameters affect the time to reach maturity, $\hat{T}$.

Proposition 4. *Given the entrepreneur's initial utility level $\underline{u}$, we have:*

(a) $\hat{T}$ strictly increases with θ and μ;

(b) $\hat{T}$ strictly decreases with r and δ;

(c) if $\alpha > r + \delta + \theta (\beta - r)$, $\hat{T}$ strictly increases with α.

Intuitively, a higher level of θ makes the enforcement of the contract more limited. A larger value for μ demands a greater rate of capital growth, which slows down the relaxation of the borrowing constraint.[9] A lower user's cost, $r + \delta$, or a larger α implies a greater borrowing capacity, which requires a greater entrepreneur's stake for maturity. All of the above delay the maturity of a firm. The following result describes the long-run Tobin's Q and the leverage rate of mature firms.

Proposition 5. *In the long run, out of the borrowing constraint, the Tobin's Q is*[10]

$$q(\hat{u}) = \frac{(1-\alpha)(r+\delta) + \theta(\beta - r)}{\alpha(r-\mu)}, \tag{16}$$

and the leverage rate is

$$l(\hat{u}) = \frac{\alpha((r-\mu) - \theta(\beta-\mu))}{(1-\alpha)(r+\delta) + \theta(\beta - r)}. \tag{17}$$

Both $q(\hat{u})$ and $l(\hat{u})$ are positive. Furthermore,

(a) $q(\hat{u})$ decreases with α, and increases with δ, θ, β, and μ;

(b) $l(\hat{u})$ increases with α, and decreases with δ, θ, β, and μ.

[9] See equation (8).

[10] Notice that $v(\hat{u}) = \underline{v}(\hat{u})$. So (10) and (11) imply an expression of $v(\hat{u})$ in parameters, which, along with (15), imply (16) and (17). Proving the rest of the proposition is straightforward.

This proposition suggests that firms have a higher Tobin's Q and are less leveraged in the long run after maturity if they are in industries with (1) more decreasing returns to scale technology, (2) a higher user's cost of capital, (3) more limited contract enforcement, or (4) higher productivity growth.

References

Rui Albuquerque and Hugo Hopenhayn. Optimal lending contracts and firm dynamics. *Review of Economic Studies*, 71:285–315, 2004.

Peter DeMarzo and Yuliy Sannikov. Optimal security design and dynamic capital structure in a continuous-time agency model. *Journal of Finance*, 61:2681–2724, 2006.

Avinash Dixit. *The Art of Smooth Pasting.* Routledge, 1999.

Zhiguo He. Optimal executive compensation when firm size follows geometric brownian motion. *Review of Financial Studies*, 22:859–892, 2009.

Yuliy Sannikov. A continuous-time version of the principal-agent problem. *Review of Economic Studies*, 75:957–984, 2008.

A Proof of Proposition 3

Part (a) is straightforward and we only show Parts (b) and (c) with $t \in \left[0, \hat{T}\right)$. Let's start with (b). According to the optimal policies,

$$w(u_t) = E_t\left[\int_0^{\hat{T}-t} e^{-(r-\mu)s}\left[\left(\frac{u_{t+s}}{\theta}\right)^\alpha - (r+\delta)\frac{u_{t+s}}{\theta}\right]ds + \int_{\hat{T}-t}^{\infty} e^{-(r-\mu)s}\hat{\pi}ds\right]$$

and

$$\begin{aligned} q(u_t) &= \frac{\theta}{u_t}E_t\left[\int_0^{\hat{T}-t} e^{-(r-\mu)s}\left[\left(\frac{u_{t+s}}{\theta}\right)^\alpha - (r+\delta)\frac{u_{t+s}}{\theta}\right]ds + \int_{\hat{T}-t}^{\infty} e^{-(r-\mu)s}\hat{\pi}ds\right] \\ &= E_t\left[\theta\int_0^{\hat{T}-t}\left[\frac{1}{\theta^\alpha}e^{\alpha(\beta-\mu)s}u_t^{\alpha-1} - \frac{r+\delta}{\theta}e^{(\beta-\mu)s}\right]ds + \frac{\theta}{u_t}\int_{\hat{T}-t}^{\infty} e^{-(r-\mu)s}\hat{\pi}ds\right]. \end{aligned}$$

The second equality above is due to $u_{t+s} = u_t e^{(\beta-\mu)s}$. So,

$$\frac{dq(u_t)}{dt} = E_t\left[\begin{array}{c} -\frac{\theta}{u_t}e^{(r-\mu)\left(\hat{T}-t\right)}\left[\left(\frac{u_{\hat{T}}}{\theta}\right)^\alpha - (r+\delta)\left(\frac{u_{\hat{T}}}{\theta}\right)\right] + \frac{\theta}{u_t}e^{(r-\mu)\left(\hat{T}-t\right)}\hat{\pi} \\ +\theta\int_0^{\hat{T}-t}\frac{d}{dt}\left[\frac{1}{\theta^\alpha}e^{\alpha(\beta-\mu)s}u_t^{\alpha-1} - \frac{r+\delta}{\theta}e^{(\beta-\mu)s}\right]ds \\ -\frac{\theta}{u_t^2}\frac{du_t}{dt}\int_{\hat{T}-t}^{\infty}e^{-(r-\mu)s}\hat{\pi}ds \end{array}\right]$$

Inside the parentheses, the two terms in the first row above correspond to the derivatives of the integral boundaries, which are canceled out. The third row is negative, and the integrand on the second row is

$$\frac{\alpha-1}{\theta^{\alpha}}e^{\alpha(\beta-\mu)s}u_t^{\alpha-2}\frac{du_t}{dt}<0,$$

and we have Part (b). To show Part (c), notice that $l\left(u_t\right)=\frac{1}{q(u_t)}-\frac{y(u_t)}{w(u_t)}$. Obviously, Part (b) implies that the first term decreases with t. The second term can be written as

$$\begin{aligned}-\frac{y(u)}{w(u)} &= \frac{-C^{\Delta}u^{\frac{r-\mu}{\beta-\mu}}}{\frac{r+\delta}{\beta-r}\left(\frac{u}{\theta}\right)+\frac{1}{r-\mu-\alpha(\beta-\mu)}\left(\frac{u}{\theta}\right)^{\alpha}+C^{w}u^{\frac{r-\mu}{\beta-\mu}}} \\ &= \frac{-C^{\Delta}}{\frac{r+\delta}{\theta(\beta-r)}u^{1-\frac{r-\mu}{\beta-\mu}}+\frac{1}{\theta^{\alpha}(r-\mu-\alpha(\beta-\mu))}u^{\alpha-\frac{r-\mu}{\beta-\mu}}+C^{w}}\end{aligned}$$

with $C^{\Delta}=C^{w}-C^{v}$, which is strictly positive. The derivative with respect to u of the denominator above is $\frac{r+\delta}{\theta(\beta-\mu)}u^{-\frac{r-\mu}{\beta-\mu}}+\frac{1}{\theta^{\alpha}}\left(\beta-\mu\right)u^{\alpha-\frac{r-\mu}{\beta-\mu}-1}$, which is strictly positive according to Assumption 1. Therefore, $-\frac{y(u_t)}{w(u_t)}$ increases with u and t, and we have the desired result.