# Appendices for Reviewers to "A tractable model of limited enforcement and the life-cycle dynamics of firms"

## A Proof of Lemma 1

Fixing $Z_0$, suppose $(\{K_{1,t}\}, \{C_{1,t}\})$ and $(\{K_{2,t}\}, \{C_{2,t}\})$ are the two optimal contracts which promise the entrepreneur the initial utility levels, $\underline{U}_1$ and $\underline{U}_2$, respectively. Obviously, for any $\eta \in (0,1)$,

$$(\{\eta K_{1,t} + (1-\eta) K_{2,t}\}, \{\eta C_{1,t} + (1-\eta) C_{2,t}\})$$

is a contract that promises the entrepreneur $\eta \underline{U}_1 + (1-\eta) \underline{U}_2$ and satisfies the borrowing constraint (4). On the other hand, the operating profit is concave in $K$, hence this contract yields an expected payoff of the investor no less than $\eta V(Z_0, \underline{U}_1) + (1-\eta) V(Z_0, \underline{U}_2)$ and we have the desired result.

## B Proof of Proposition 1

The main task in the proof is to verify the optimality of the optimal contract and the associated normalized value function. It is easy to show the following preliminary result.

**Lemma B.1.** *$v(\cdot)$ satisfies HJB (9); $v''(u) \le 0$ and $v'(u) \ge -1$; the inequalities are strict over $[0, \hat{u})$ and binding for $u \ge \hat{u}$.*

Verification of the optimality of the contract is divided in to Lemmas B.2 and B.3.

**Lemma B.2.** *The contract characterized in Proposition 1 generates the normalized value function $v(u)$.*

*Proof.* Suppose $\underline{u} = \underline{u} \in [0, \hat{u}]$, and let $\hat{v}_t$ be the normalized firm value to the investor at $t$ under the contract and $\hat{T}$ be the time such that $u_{\hat{T}} = \hat{u}$ under the contract. Then for $t \ge \hat{T}$, $k_t = \frac{\hat{u}}{\theta}$ and $dC_t = Z_t(\beta - \mu)\hat{u}dt$. Then the normalized payoff to the investor at $\hat{T}$

is

$$\begin{aligned}
\hat{v}_{\hat{T}} &= \frac{1}{Z_{\hat{T}}} E_{\hat{T}}\left[\int_{\hat{T}}^{\infty} e^{-r\left(t-\hat{T}\right)} Z_t\left(\left(\frac{\hat{u}}{\theta}\right)^{\alpha}-(r+\delta)\,\frac{\hat{u}}{\theta}-(\beta-\mu)\,\hat{u}\right) dt\right] \\
&= \frac{1}{r-\mu}\left[\left(\frac{\hat{u}}{\theta}\right)^{\alpha}-(r+\delta)\,\frac{\hat{u}}{\theta}-(\beta-\mu)\,\hat{u}\right] \qquad \text{(B.1)}
\end{aligned}$$

Since $v'(\hat{u}) = -1$, the HJB equation (9), definition of $\hat{u}$, (11), and (B.1) imply $v(\hat{u}) = \hat{v}_{\hat{T}}$. Now, for $t \leq \hat{T}$, we define

$$\Psi_t \equiv \int_0^t e^{-rs}\left(\left(Z_s^{1-\alpha}K_s^{\alpha}-(r+\delta)\,K_s\right) ds - dC_t\right) + e^{-rt} Z_t v\left(u_t\right).$$

Obviously, $\Psi_{\hat{T}} = Z_0\hat{v}_0$, the normalized payoff to the investor at $t = 0$, and

$$e^{rt} d\Psi_t = Z_t\left[k_t^{\alpha} - (r+\delta)\,k_t - (r-\mu)\,v\left(u\right) + (\beta-\mu)\,u_t v'\left(u_t\right)\right] dt$$

Since $\hat{u} < u^*$, $k_t = \frac{u_t}{\theta}$, the optimal policy implied by the maximization problem on the right hand side of (9). Therefore, (9) implies that $\{\Psi_t\}$ is a super martingale and

$$v\left(\underline{u}\right) = \frac{1}{Z_0}\Psi_0 = \frac{1}{Z_0} E_0\left[\Psi_{\hat{T}}\right] = \hat{v}_0$$

and we have the desired result. □

**Lemma B.3.** *The contract characterized in Proposition 1 is optimal*

*Proof.* We show that the normalized payoff of the investor under any contract satisfying the constraint (4) and promising the entrepreneur $\underline{u} > 0$ is no larger than $v\left(\underline{u}\right)$. Let $\left(\left\{\tilde{C}_t\right\}, \left\{\tilde{K}_t\right\}\right)$ be such an alternative contract which implies the entrepreneur's continuation-utility process $\left\{\tilde{U}_t\right\}$. We accordingly define $\tilde{u}_t$, $\tilde{k}_t$, $\tilde{g}_t$ and the investor's normalized payoff process $\tilde{v}_t$. Let $\tilde{T} = \inf\{t \geq 0 : \tilde{u}_t = 0\}$. Then (4) implies that $\tilde{u}_t = 0$, $\tilde{k} = 0$ and $\tilde{v}_t = 0$ for $t \geq \tilde{T}$. So $\tilde{v}_{\tilde{T}} = v(0)$.[1] For $t < T$ we define

$$\tilde{\Psi}_t \equiv \int_0^t e^{-rs}\left[Z_s\left(\tilde{k}_s^{\alpha} - (r+\delta)\,\tilde{k}_s\right) ds - d\tilde{C}_s\right] + e^{-rt} Z_t v\left(\tilde{u}_t\right).$$

So $\tilde{\Psi}$ is the expected payoff of the investor if she follow the alternative contract up to $t$ and then switches to the contract we described in the proposition. Obviously, $\tilde{\Psi}_0 = Z_0 v\left(v_0\right)$ and $\tilde{\Psi}_{\tilde{T}}$ is the expected payoff of the investor under the alternative contract. Hence

$$e^{-rt} d\tilde{\Psi}_t = Z_t\left\{\begin{array}{c}\left[\tilde{k}_t^{\alpha} - (r+\delta)\,\tilde{k}_t - (r-\mu)\,v\left(\tilde{u}_t\right) + (\beta-\mu)\,\tilde{u}_t v'\left(\tilde{u}_t\right)\right] dt \\ -\left(1+v'\left(\tilde{u}_t\right)\right)\frac{1}{Z_t} d\tilde{C}_t\end{array}\right\} \qquad \text{(B.2)}$$

[1]Notice that $\tilde{T} = \infty$ if $\tilde{u}_t$ never hits zero under the contract.

Since $v$ satisfies the HJB equation (2), the coefficient of $dt$ in the first row is non-positive; the coefficient of $d\tilde{C}_t$ is non-positive because $v'(\tilde{u}) \geq -1$(Lemma B.1). Therefore $\left\{\tilde{\Psi}_t\right\}$ is a super martingale and

$$v\left(v_0\right) = \frac{1}{Z_0}\tilde{\Psi}_0 \geq \frac{1}{Z_0}E_0\left[\tilde{\Psi}_{\tilde{T}}\right].$$

So we have the desired result. □

# C Proof of Proposition 2

Given the policy functions under the optimal contract, for $u \in [0, \hat{u}]$, $w(u)$ satisfies the following HJB differential equation.

$$0 = \left(\frac{u}{\theta}\right)^{\alpha} - (r+\delta)\frac{u}{\theta} - (r-\mu)\,w\left(u\right) + (\beta-\mu)\,uw'\left(u\right).$$

Obviously, on the left boundary, $w(0) = 0$; on the right boundary, when $\hat{u}$ is reached, $k_t = \hat{k} = \frac{\hat{u}}{\theta}$ which is time invariant. Therefore $w\left(\hat{u}\right) = \frac{\hat{\pi}}{r-\mu}$. So $w$ is characterized by (14) and (14).

Now we show that $w(u)$ increases with $u$. Notice that we only need to show the result over $[0, \hat{u}]$. Suppose that two optimal contracts start with two different initial utility levels promised to the entrepreneur, $\underline{u}^1$ and $\underline{u}^2$, with $\underline{u}^1 < \underline{u}^2 < \hat{u}$. Denote the $\{u_t\}$-process under the two contracts $\{u_t^1\}$ and $\{u_t^2\}$ respectively. According to the policy functions characterized in Proposition 1, $k\left(u_t^1\right) \leq k\left(u_t^2\right) < \hat{k} < k^*$ and then

$$\pi\left(k\left(u_t^1\right)\right) \leq \pi\left(k\left(u_t^2\right)\right) \text{ for all } t \geq 0$$

with the inequality being strict before $u_t^1$ hits $\hat{u}$. Here $\pi(k) = k^{\alpha} - (r+\delta)\,k$. Therefore

$$w\left(\underline{u}^1\right) - w\left(\underline{u}^2\right) = E_0\left[\int_0^{\infty} e^{-rt}Z_t\left(\pi\left(k\left(u_t^1\right)\right) - \pi\left(k\left(u_t^2\right)\right)\right)dt\right] < 0.$$

So we have $w'(u) \geq 0$ and the inequality is strict if $u < \hat{u}$.

# D Proof of Proposition 4

We focus on Parts (a) and (c) as the results in Part (b) are straightforward.

Part (a): It is easy to show result about $\mu$ and we show the one about $\theta$. According to the definition of $\hat{T}$, we need to show that $\hat{u}$ strictly increases with $\theta$. Notice that

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \theta} &= \left[\frac{\alpha}{r+\delta+\theta(\beta-r)}\right]^{\frac{1}{1-\alpha}} + \left[\frac{\alpha}{r+\delta+\theta(\beta-r)}\right]^{\frac{\alpha}{1-\alpha}} \frac{-\alpha(\beta-r)}{(r+\delta+\theta(\beta-r))^2} \\ &= \left[\frac{\alpha}{r+\delta+\theta(\beta-r)}\right]^{\frac{1}{1-\alpha}} \frac{r+\delta}{r+\delta+\theta(\beta-r)} \\ &> 0.\end{aligned}$$

So we have the desired result.

Part (c): We need to show that $\hat{u}$ strictly increases with $\alpha$ when the condition is satisfied. According to (11),

$$\begin{aligned}\frac{\partial \hat{u}}{\partial \alpha} &= \frac{\theta}{1-\alpha}\left[\frac{\alpha}{r+\delta+\theta(\beta-r)}\right]^{\frac{1}{1-\alpha}} \\ &\quad +\theta \ln\left[\frac{\alpha}{r+\delta+\theta(\beta-r)}\right]\left[\frac{\alpha}{r+\delta+\theta(\beta-r)}\right]^{\frac{1}{1-\alpha}} \frac{1}{(1-\alpha)^2}.\end{aligned}$$

The first term on the right hand side is positive, and so is the second if $\frac{\alpha}{r+\delta+\theta(\beta-r)} > 1$. Hence, we have the desired result.

# E Proof of Proposition 5

According to (15)

$$q(\hat{u}) = \frac{\hat{k}}{w(\hat{u})} = \left(\frac{\hat{\pi}}{r-\mu}\right) / \left(\frac{\hat{u}}{\theta}\right).$$

By plugging in (11), we have (16). The value matching conditions (13) and (15) imply

$$y(\hat{u}) = w(\hat{u}) - v(\hat{u}) = \frac{\beta-\mu}{r-\mu}\hat{u}.$$

Thus

$$l(\hat{u}) = \left(\left(\frac{\hat{u}}{\theta}\right) - \frac{\beta-\mu}{r-\mu}\hat{u}\right) / \left(\frac{\hat{\pi}}{r-\mu}\right).$$

By plugging in (11), we have (17). It is easy to check that $q(\hat{u})$ is positive and Assumption 1 guarantees the positivity of $l(\hat{u})$. The dependences of $q(\hat{u})$ and $l(\hat{u})$ on the parameters are obvious.