Information-Driven Volatility

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Abstract: Modern asset pricing theory predicts an unambiguously positive relationship between volatility and expected returns. Empirically, however, *realized* volatility in the past often predicts expected returns in the future with a negative sign, as exemplified by the volatility-managed portfolios of Moreira and Muir (2017). Theoretically, we show that information-driven volatility induces a negatively correlation between past realized volatility and expected volatility and expected returns in the future. We develop a simple asset pricing model based on this intuition and demonstrate that our model can account for several volatility-related asset pricing puzzles such as the return on volatility managed portfolios, the "variance risk premium" return predictability (Bollerslev, Tauchen, and Zhou, 2009), and the predictability of returns by implied volatility reduction on macroeconomic announcement days.

Keywords: Volatility Managed Portfolios, Variance Risk Premium, Macroeconomic Announcements, Generalized Risk Sensitivity

JEL Code: D83, D84, G11, G12, G14

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1 Introduction

The relationship between volatility and expected returns, or the risk and return trade-off is a fundamental concept in finance. Most of the consumption-based asset pricing models of time-varying volatility assume stochastic volatility of macroeconomic fundamentals, such as stochastic volatility in aggregate consumption or aggregate productivity. However, several facts about volatility and the relationship between volatility and expected returns on financial markets prove to be puzzling from this traditional viewpoint of stochastic volatility. First, financial market volatility exhibits lots of high-frequency variations, while changes in the volatility of macroeconomic fundamentals happen at much lower frequencies, such as at the annual or the decade frequency. Second, while standard stochastic volatility models often predict a robust positive relationship between past realized volatility and future expected returns, many researchers have documented a negative relationship between past realized volatility and future expected returns, for example, Nelson (1991) and Glosten, Jagannathan, and Runkle (1993). More recently, Moreira and Muir (2017) construct a volatility managed portfolios that take less risk when realized volatility is high and more risk when realized volatility is low. They show that the volatility managed portfolio produces large excess returns relative to the market. Third, although neither implied nor realized volatility has strong predictive powers for returns, the difference between the two, often called the variance risk premium, has strong predictive powers for returns over the three-to-six-month horizon.

In this paper, we develop an asset pricing model where stochastic volatility originates not from changes in the volatility of macroeconomic fundamentals but variations of the informativeness of publicly available information events, such as macroeconomic announcements. While empirical evidences for changes in the volatility of macroeconomic fundamentals are mostly about long-run trends that happen at very low frequencies, information continuously arrives at financial markets and affect the volatility of asset prices at the daily, hourly, or even higher frequencies. The contrast between our theory of information driven volatility and the standard models of stochastic volatility is best illustrated using the following variance decomposition identity:

$$Var\left[C_T\right] = Var\left[E\left(C_T|X_t\right)\right] + E[Var\left(C_T|X_t\right)]. \quad (1)$$

We interpret C_T as macroeconomic fundamentals such as aggregate consumption, the value of which will be realized at the terminal time T. We interpret X_t as a public signal revealed at time $t < T$ that are informative about C_T. One example for such public signals is a macroeconomic announcement. To fix ideas, we interpret X_t as a macroeconomic announcement, the above formula then decomposes the total variance of macroeconomic fundamentals, $Var\left[C_T\right]$ into a variance realized on the announcement day t, $Var\left[E\left(C_T|X_t\right)\right]$, and a variance that will realize after t, $E\left[Var\left(C_T|X_t\right)\right]$. Traditional models of stochastic volatility assume time-variations in the variance of macroeconomic fundamentals, $Var\left[C_T\right]$, while our theory focuses on variations in the informativeness of macroeconomic news, that is, the $Var\left[E\left(C_T|X_t\right)\right]$ term.

From a theoretical perspective, changes in $Var\left[C_T\right]$ affect the total quantity of risk. Because volatility shocks are typically persistent, a high realization of $Var\left[C_T\right]$ in standard models also forecasts a high expected volatility in the future and therefore implies a high expected return going forward. However, this is precisely where the traditional theory has difficulty, as empirical evidence is in favor of a negative relationship between past realized volatility and future expected returns at relatively high frequencies. In contrast, changes in $Var\left[E\left(C_T|X_t\right)\right]$ affect the intertemporal distribution of risk and risk compensation. Holding the total amount of risk $Var\left[C_T\right]$ constant, a higher realization of $Var\left[E\left(C_T|X_t\right)\right]$ is associated with a larger realization of the risk premium on announcement days, but also implies that on average, there will be a lower quantity of risk, $Var\left(C_T|X_t\right)$ in the future. We show that if preferences satisfy the property of generalized risk sensitivity of Ai and Bansal (2018), then lower expected variance $E\left[Var\left(C_T|X_t\right)\right]$ is also associated with a lower expected return in the future, giving rise to a negative correlation between past realized volatility and future expected returns, as documented by the previous empirical literature.

There are two motivations for our focus on information driven volatility. First, the documented negative relationship between realized volatility and future expected returns are mostly at monthly or higher frequencies. The volatility managed portfolios of Moreira and Muir (2017) re-balances every month, and the predictability evidence of Nelson (1991) uses daily returns. Empirically, variations in the volatility of macroeconomic fundamentals operate at a much lower frequency and are unlikely to affect the day-to-day volatility of stock market returns in significant ways. At high frequencies such as monthly and daily frequencies, keeping the variance of macroeconomic fundamentals, $Var\left[C_T\right]$, fixed allows us to focus on the impact of information captured by the term $Var\left[E\left(C_T|X_t\right)\right]$.

Second, the empirical evidence of the macroeconomic announcement premium implies that investor preferences must satisfy generalized risk sensitivity (Ai and Bansal (2018)). Our theory of information driven volatility relies on generalized risk sensitivity, because as we show in the paper, this condition allows information driven volatility to require a risk premium in equilibrium and predicts future returns. The recent empirical literature have convincingly established the importance of information in determining the stock market risk compensation. For example, Savor and Wilson (2013, 2014) demonstrate that more than 60% of the equity premium is realized on a small number of macroeconomic announcement days. Lucca and Moench (2015) emphasize the importance of FOMC announcements in risk compensation. Ai and Bansal (2018) demonstrate that the empirical evidence of the macroeconomic announcement premium implies that the preference for the representative consumer must satisfy generalized risk sensitivity. Our theory that links risk compensation to information driven volatility relies on the condition of generalized risk sensitivity developed in Ai and Bansal (2018), which is supported empirically by the literature on macroeconomic announcement premium.

To assess the quantitative importance of the mechanism of information-driven volatility, we develop a continuous-time asset pricing model with homoscedasticity but with time-varying information quality. In our model, the volatility of aggregate consumption is constant, and yet the financial market exhibits stochastic volatility because the informativeness of public signals is time varying.

First, we conduct several empirical tests for the unique implications of the mechanism of information driven volatility using both the U.S. stock market return data and the model simulated data. In the data, FOMC announcements are the most identifiable events that reveal information about the macroeconomy. To test the mechanism of information driven volatility, we first develop a measure of informativeness of FOMC announcements. Using this measure, we show that more informative macroeconomic announcements are associated with high realized returns and high implied volatility reduction upon announcements, and low realized returns and low realized volatility post announcements both in the data and in the model.

Second, we construct the volatility managed portfolio in our model and replicate the Moreira and Muir (2017) exercise in our model. Following Moreira and Muir (2017), we construct a portfolio in our model that takes more leverage to invest in the market portfolio when past realized volatility is low and reduces leverage and invests more in the risk-free bond when market volatility is high. We show that similar to the evidence documented in the Moreira and Muir (2017) paper, in our model, the volatility managed portfolio earns a higher return than the market portfolio.

In addition, we replicate the variance risk premium predictability exercise of Bollerslev, Tauchen, and Zhou (2009) in our calibrated model. Despite the absence of stochastic volatility in macroeconomic fundamentals, the difference between the implied and the past realized volatility predicts stock market returns in our model. Because implied volatility is a forwarding looking measure of volatility, the difference between the implied and past realized volatility reflects the informativeness of the upcoming information event. It predicts returns because, holding the past realized volatility constant, the release of a highly informative signal is associated with both a high implied volatility before the event and the realization of larger risk compensation associated with the event. If the upcoming public signals are expected to be informative, the implied variance will rise, but the realized variance stays the same. The empirically documented variance risk premium predictability can therefore be explained by the information channel in our model without assuming high-frequency movements in macroeconomic volatility.

We assume homoscedastic shocks in macroeconomic fundamentals in our model to emphasize the importance of the information driven volatility. However, we do not intend to argue that time-varying macroeconomic volatility is absent or unimportant for understanding equity market risk compensations. Our purpose is to distinguish two notions of uncertainty: the variance of macroeconomic fundamentals and the variance of an investor's posterior beliefs. We argue that variations in the posterior beliefs affect the intertemporal distribution of risk and risk compensation and are more important in understanding short-horizon risk compensations in financial markets. Changes in the variance of macroeconomic fundamentals affect the total quantity of risk will undoubtedly have prominent impacts on risk and risk compensation, but the effects are likely to manifest only over longer horizons.

Related literature Our model builds on the literature on learning and information in financial markets. David (1997) and David (2008) develop learning models to study equity market risk compensations. Veronesi (2000) and Ai (2010) study how information quality affects the aggregate stock market risk premium. David and Veronesi (2013) estimate a regime-switching model with learning. Pastor and Veronesi (2009b) develop a learning model to study the relationship between technological innovations and stock market valuations. Pastor and Veronesi (2009a) provide an excellent review of the literature on learning and asset pricing. Bansal and Shaliastovich (2010, 2011) and Shaliastovich (2015) develop models where learning results in asset price jumps. None of the above papers focus on time-varying informativeness of macroeconomic news as we do.

Our paper is evidently related to the vast empirical literature on the expected return-volatility relationship. Both Nelson (1991) and Glosten, Jagannathan, and Runkle (1993) document empirical evidence that is supportive of a negative relationship between past realized volatility and future expected returns. Harvey (1989) finds mixed evidence or a time-varying relationship between expected excess returns and conditional variances. Consistent with our theory, Harrison and Zhang (1999) find a negative relationship and sometimes a mixed evidence for the relationship between past realized volatility and future expected returns over short horizons, but a positive relationship over horizons longer than a year. More recently, Moreira and Muir (2017) demonstrate a positive average return on their volatility managed portfolio relative to the market return, and their evidence is also consistent with a negative relationship between past realized volatility and future expected returns. Lochstoer and Muir (forthcoming) develop a model of extrapolative expectations of volatility shocks to explain the variance risk premium predictability and the negative relationship between volatility and expected returns.

Several recent papers provide empirical evidence that is consistent with the information driven volatility channel emphasized in this paper. Baker, Bloom, Davis, Kost, Sammon, and Viratyosin (2020) document that a higher clarity of news is associated with lower realized volatility in the future. Zhang and Zhao (2020) provide evidence that in periods where public information is imprecise, the realized macroeconomic announcement premium is low. Chaudhry (2021) shows that announce days are typically associated with uncertainty reductions and lower expectations post announcements.

Our paper is also related to the variance risk premium predictability literature. Bollerslev, Tauchen, and Zhou (2009) document the predictability of stock market returns by the difference between implied and realized variance, and develop a model of variance risk premium predictability based on stochastic volatility in the volatility of macroeconomic fundamentals.Drechsler and Yaron

(2011) develop a model with stochastic volatility and stochastic jumps to quantitatively explain the variance risk premium predictability. Eraker and Wang (2015) estimate a non-linear diffusion model and study the variance risk premium predictability. Zhou (2018) provides a thorough review of this literature. The above literature has interpreted the difference between implied and realized variance as the difference between variance under the physical measure and that under the risk neutral measure and hence defined as the variance risk premium. We show that the difference between implied and realized variance can predict returns without assuming a variance risk premium. In our model, the difference between the two reflects the informativeness of the upcoming announcement. It predicts returns because in our model, resolution of uncertainty is associated with realizations of risk premium.

This paper is also closely related to the literature on generalized risk sensitivity and macroeconomic announcements. From the empirical perspective, pre-scheduled macroeconomic announcements are the most salient information events that are associated with significant realizations of market equity premiums and affect the volatility of stock market returns. The literature that documents a significant macroeconomic announcement premium, for example, Savor and Wilson (2013, 2014) and Lucca and Moench (2015) provide strong empirical support for the mechanism emphasized in this paper. From the theoretical point of view, Ai and Bansal (2018) demonstrate that the existence of announcement premium implies generalized risk sensitivity in preferences. In our setup, as we will demonstrate in Section 3 of the paper, generalized risk sensitivity is also necessary for information quality to affect the intertemporal distribution of risk compensation.

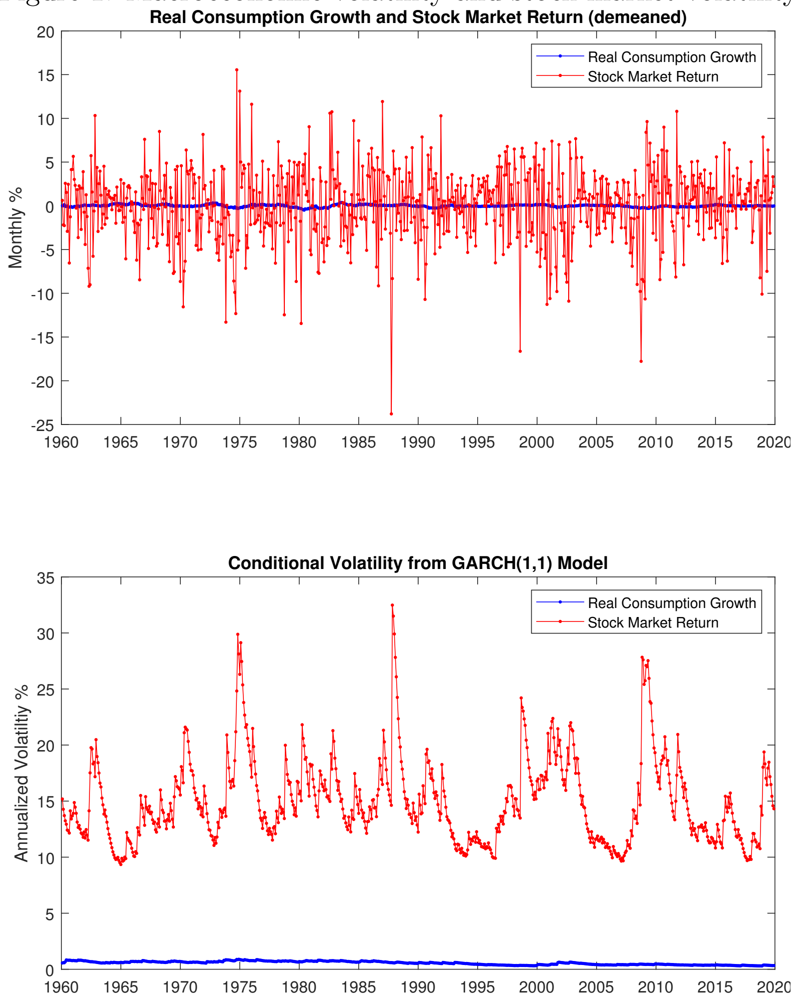
The rest of the paper is organized as follows. In Section 2 summarizes the stylized facts between realized variance and expected returns. In Section 3presents a simple two-period model to illustrate the impact of informativeness of macroeconomic news on the intertemporal distribution of risk and risk compensation. Section 4 develops a dynamic model to account for the stylized facts, and Section 5 presents the quantitative results. Section 6 concludes.

2 Stylized Facts

In this section, we provide details on several stylized facts on volatility and stock market returns that motivate the development of our theory.

1. The volatility of macroeconomic fundamentals do not exhibit significant variations at the monthly or annual frequency.

Figure 1: Macroeconomic volatility and stock market volatility



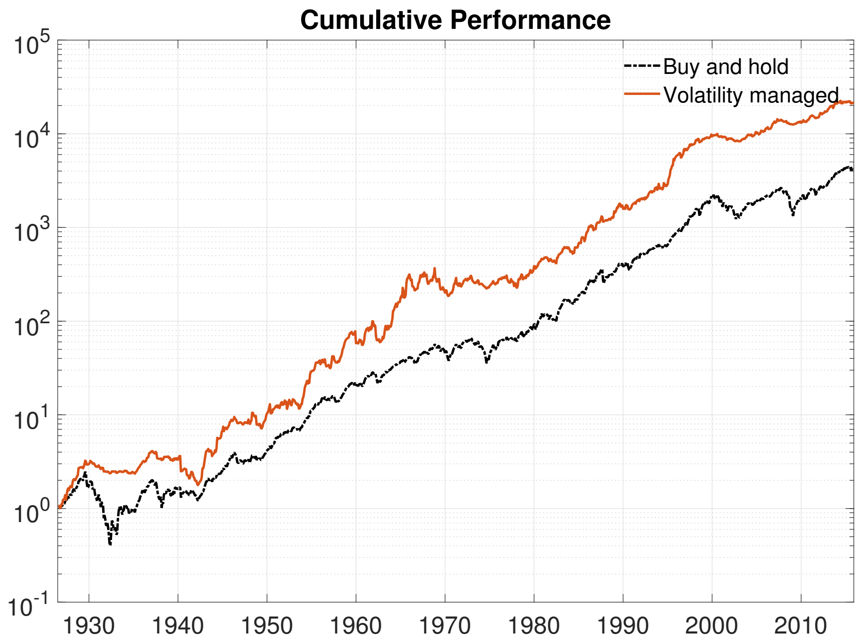
The top panel is the monthly consumption growth rates (solid line) and the S&P500 index returns (dash-dotted line) during the period of 1960.02-2019.12. The bottom panel is the estimated conditional volatility of the two series from a GARCH(1,1) model during the same sample period.

In Figure 1, we plot the time series of monthly consumption growth and monthly stock market returns in the top panel. In the bottom panel, we plot the estimated conditional volatility of the two time series from a GARCH (1,1) model. Compared to stock market returns, the variations of consumption growth are much smaller. The estimated conditional volatility of stock returns exhibits sharp variations over the monthly horizon, while that of aggregate consumption growth is virtually flat by comparison.

2. Strategies that take more leverage when volatility is low and take less market risk exposure

when volatility is high produce large alphas. This is contradictory to the positive relationship between past realized volatility and future expected returns predicted by standard asset pricing models with stochastic volatility.

Figure 2: The volatility managed portfolio



This figure plots the return on a volatility managed portfolio (solid line) and the return on the buy-and-hold market portfolio (dashed line) during the period of 1926.07-2015.12.

In Figure 2, we follow Moreira and Muir (2017) and construct a volatility managed portfolio that is rebalanced every month according to past-month realized volatility. Consistent with their result, we find that a volatility managed market portfolio produces an average annual return of 9.54% per year from 1926 through 2015, while the average market return on a buy-and-hold strategy is 7.75% per year during the same period.

3. Even though realized market variance or implied variance do not have strong predictive powers for returns, the difference between the two does for returns over three-to-six month horizons. This is the well-known variance risk premium predictability (Bollerslev, Tauchen, and Zhou (2009)). In Table 1, we report results of the following standard VRP predictability regression:

$$R_{t,t+\Delta} = \alpha + \beta \left[IV_t - RV_{t-21,t} \right] + \varepsilon_{t,t+\Delta}, \tag{2}$$

where $R_{t,t+\Delta}$ is the cumulative market return from time t to time $t+\Delta$, where $\Delta = 21, 42, 63, 84, 105, 126$ trading days. IV_t is the forward-looking 30-day implied variance (VIX index squared) at time t, and $RV_{t-21,t}$ is the past 30-day realized variance. The regression coefficients are statistically significant and increasing up to six months.

Table 1: Return predictability by $IV - RV$

Number of days	21	42	63	84	105	126
$IV_t - RV_t$	0.03	0.04	0.08	0.13	0.10	0.08
	(2.95)	(1.67)	(2.83)	(5.59)	(2.57)	(1.87)
R^2 (%)	1.62	1.81	4.47	9.39	3.99	2.34

This table presents the results of the return predictability regression (2). Columns 2 to 7 represent returns on the left hand side of (2) with $\Delta = [21, 42, 63, 84, 105, 126]$ trading days. The regression includes returns and VRPs every 21 days during the period of 1990.01.01-2019.12.31. Newey-West t-statistics with 1-6 lags are in parentheses.

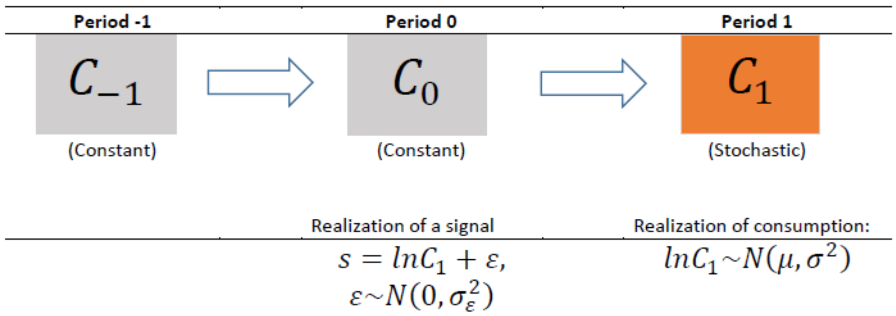
3 A Three-Period Model

In this section, we present a three-period model to illustrate the mechanism of information driven volatility. We show how the informativeness of public signals affects the intertemporal distribution of risk and, under the condition of generalized risk sensitivity, also affects the intertemporal distribution of risk compensation.

We consider a representative-agent economy with three periods -1, 0, and 1. We use C_t to denote aggregate consumption in period t, and we plot the event tree for the economy in Figure 3. For simplicity, we assume that both C_{-1} and C_0 are deterministic. Consumption in period 1, C_1, follows a lognormal distribution with $\ln C_1 \sim \mathcal{N}\left(\mu, \sigma^2\right)$. At time 0, there is a public announcement that provides a noisy signal about $\ln C_1$ of the form $s = \ln C_1 + \varepsilon$ with $Var\left[\varepsilon\right] = \sigma_\varepsilon^2$.

In the lognormal setup, we can explicitly compute $Var\left[E\left[\ln C_1 | s\right]\right]$ and $E\left[Var\left[\ln C_1 | s\right]\right]$ in the variance decomposition formula of Equation (1). The expressions for the stochastic discount factors are also standard: $\Lambda_{-1,0} = \frac{\beta}{1-\beta}\left(\frac{C_0}{C_{-1}}\right)^{-1}\left(\frac{V_0}{m_{-1}}\right)^{1-\gamma}$ and $\Lambda_{0,1} = \frac{\beta}{1-\beta}\left(\frac{C_1}{C_0}\right)^{-1}\left(\frac{C_1}{m_0}\right)^{1-\gamma}$, where $\Lambda_{t,t+1}$ denotes the stochastic discount factor that prices date-$t+1$ consumption goods in terms of date-t consumption goods. Here, we use $V_0 = C_0^{1-\beta}\left(E\left[C_1^{1-\gamma}\right]\right)^{\frac{\beta}{1-\gamma}}$ for the date-0 utility of the agent. $m_0 = \left(E\left[C_1^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}$ is the date-0 certainty equivalent of future utility and $m_{-1} = \left(E\left[V_0^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}$ is the date--1 certainty equivalent of future utility. The following proposition

Figure 3: A three-period model



This figure illustrates the timing of the two-period model. Consumption in period -1 and that in period 0 are constant, whereas the consumption in period 1 follows a lognormal distribution.

demonstrates how the precision of the information, σ_ε^{-2}, affects the risk the risk premium upon and after the announcement.

Proposition 1. *(Intertemporal distribution of risk and risk compensation)*

Let $\ln C_1 \sim \mathcal{N}\left(\mu, \sigma^2\right)$*, then the total variance,* $Var\left[E\left[\ln C_1|s\right]\right] + Var\left[\ln C_1|s\right] = \sigma^2$*, and the variance of the conditional expectation is:*

$$Var\left[E\left[\ln C_1|s\right]\right] = \lambda\sigma^2, \tag{3}$$

where $\lambda = \frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}$*, and* $\frac{\partial}{\partial\sigma_\varepsilon^{-2}}\lambda > 0$*.*

Furthermore, suppose $\gamma > 1$*, both* $\Lambda_{-1,0}$ *and* $\ln\Lambda_{0,1}$ *are decreasing functions of* C_1*. In addition, the total variance of the stochastic discount factor is* $Var\left[\ln\left(\Lambda_{-1,0} \times \Lambda_{0,1}\right)\right] = \gamma^2\sigma^2$*, and the variance of the announcement stochastic discount factor is*

$$Var\left[\ln\Lambda_{-1,0}\right] = \lambda\sigma^2\beta^2\left(1-\gamma\right)^2. \tag{4}$$

Equation (3) is derived from the variance decomposition formula assuming the total amount of variance is fixed. The fact that $\frac{\partial}{\partial\sigma_\varepsilon^{-2}}\lambda > 0$ implies that more precise signals are associated with a larger fraction of risk being released upon announcement at time 0. If we think of stock price as reflecting agent's belief about $\ln C_1$, then higher signal precision should be associated with larger stock market reactions upon the announcement of s. Clearly, the release of the signal s does not change the total amount of risk, but affects the intertemporal distribution of it.

The second part of the above proposition is about the risk compensation at and after the announcement. We assume $\gamma > 1$, which reflects the condition of generalized risk sensitivity of (Ai and Bansal (2018)). Under this condition, the announcement at time 0 is associated with a non-trivial volatility of the stochastic discount factor: $Var\left[\ln \Lambda_{-1,0}\right]$, and in addition, $\frac{\partial}{\partial \sigma_{\varepsilon}^{-2}} Var\left[\ln \Lambda_{-1,0}\right] > 0$, that is, higher precision of the information revealed at the announcement is associated with a higher volatility of the announcement stochastic discount factor. Because the total volatility of the stochastic discount factor, $Var\left[\ln\left(\Lambda_{-1,0} \times \Lambda_{0,1}\right)\right]$ does not depend on the precision of information, $\sigma_{\varepsilon}^{-2}$, Equation (4) implies that a higher precision of the signal corresponds to a higher risk compensation on date 0 but a lower risk compensation on date 1. Information quality affects not only the intertemporal distribution of risk but also that of the risk compensation: when public announcements are informative, the announcement premium is higher, but the risk premium going forward will be lower.

Note that if $\gamma = 1$, that is, the representative agent has an expected utility, then the precise of the signal, $\sigma_{\varepsilon}^{-2}$ will still affect $Var\left[E\left[\ln C_1 | s\right]\right]$, but not the risk compensation at the announcement, that is, $Var\left[\ln \Lambda_{-1,0}\right]$ in equation (4). The assumption $\gamma > 1$ reflects the condition of generalized risk sensitivity (Ai and Bansal (2018)). In general, under the condition of generalized risk sensitivity, the magnitude of announcement premium is increasing in the precision of signals. The recursive preference with a unit intertemporal elasticity of substitution (IES) is a special case where generalized risk sensitivity is equivalent to $\gamma > 1$.

4 A Dynamic Model

In this section, we develop a simple dynamic model with time-varying information quality. Our purpose is to use a parsimonious model to demonstrate how stochastic shocks to the quality of information change the intertemporal distribution of risk and risk compensation and allow our model to account for the stylized facts we document in Section 2. We shut down all other mechanisms for time-varying risk premium by assuming homoscedasticity in all macroeconomic fundamentals and constant elasticity of substitution (CES) preferences. We do not intend to argue that these other mechanisms are not important in driving variations in risk premiums in the data. Instead, we abstract from other mechanisms of the time-varying risk premium for two reasons. First, it allows us to highlight the mechanism of information-driven volatility. Second, we believe, at the monthly or higher frequencies, time-varying information is much more likely to affect risk premium than time-varying risk aversion or time-varying volatility, both of which are likely to vary at only lower frequencies.

Preferences and endowment We consider an endowment economy where the representative agent has a CES recursive preference with a risk aversion γ and an IES ψ. The aggregate endowment follows a diffusion process of the form:

$$\frac{dY_t}{Y_t} = \theta_t dt + \sigma_Y dB_{Y,t}, \tag{5}$$

where $B_{Y,t}$ is a standard Brownian motion and $\{\theta_t\}_{t\geq 0}$ is a two-state Markov process with the state space $\Theta = \{\theta_H, \theta_L\}$, where $\theta_H > \theta_L$. The transition matrix for θ_t over a small interval Δ is

$$\begin{bmatrix} e^{-\lambda_H \Delta} & 1 - e^{-\lambda_H \Delta} \\ 1 - e^{-\lambda_L \Delta} & e^{-\lambda_L \Delta} \end{bmatrix}.$$

where intensity λ_H is the rate of transition from high to low state, and λ_L is the rate of transition from low to high. We assume that the state variable θ_t is unobservable to investors. However, information about θ_t continuously arrives into the financial markets. Investors observe two sources of information about θ_t. First, the aggregate consumption itself contains information about θ_t. Second, pre-scheduled macroeconomic announcements are made at time $T, 2T, \ldots, nT$. We assume that announcements carry a noisy signal of θ_t. The distribution of the signal is given as follows. At the announcement at time nT, if $\theta_{nT} = \theta_H$,

$$\begin{array}{lll} s = \theta_H & with & prob\ \nu_n \\ s = \theta_L & with & prob\ 1 - \nu_n \end{array},$$

and if $\theta = \theta_L$,

$$\begin{array}{l} s = \theta_H \quad with \quad prob\ 1 - \nu_n \\ s = \theta_L \quad with \quad \quad prob\ \nu_n \end{array}.$$

Here $\nu_n \in \left[\frac{1}{2}, 1\right]$ is a parameter that measures the information quality. When $\nu_n = 1$, announcements carry perfectly accurate information, and $\nu_n = 0.5$ indicates that announcements are completely uninformative. For simplicity, we assume that $\nu_1, \nu_2, \ldots, \nu_n$ are i.i.d. over time.

Asset prices We define $\pi_t = P_t(\theta_t = \theta_H)$ as the probability of $\theta_t = \theta_H$ and $\hat{\theta}_t = E_t[\theta_t]$ to be the posterior mean of θ_t. That is, $\hat{\theta}_t = \pi_t \theta_H + (1 - \pi_t)\theta_L$. The life-time utility of the representative agent can be written as a function of state variables of the form $V\left(\hat{\theta}_t, t, Y_t\right) = H\left(\hat{\theta}_t, t\right) Y_t$. In Appendix 7.2, we show that the function $H\left(\hat{\theta}_t, t\right)$ satisfies PDEs with appropriate boundary conditions. Given the value function, we can construct the pricing kernel $M(t)$. The law of motion of $M(t)$ in the interior of $(nT, (n+1)T)$ can be written as

$$\frac{dM(t)}{M(t)} = -r\left(\hat{\theta}, t\right) dt - \sigma_M\left(\hat{\theta}, t\right) d\hat{B}_{Y,t} \tag{6}$$

where $r\left(\hat{\theta}, t\right)$ is the risk free rate:

$$\begin{aligned} r(\theta, t) = {} & \rho + \frac{1}{\psi}\theta - \frac{1}{2}\gamma\left(1 + \frac{1}{\psi}\right)\sigma_Y^2 + \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}\frac{H_\theta(\theta, t)}{H(\theta, t)}(\theta_H - \theta)(\theta - \theta_L) \\ & + \frac{\left(\frac{1}{\psi} - \gamma\right)\left(1 - \frac{1}{\psi}\right)}{2(1-\gamma)^2}\left(\frac{H_\theta(\theta, t)}{H(\theta, t)}\right)^2 \frac{(\theta_H - \theta)^2(\theta - \theta_L)^2}{\sigma_Y^2}, \end{aligned} \tag{7}$$

and $\sigma_M\left(\hat{\theta}, t\right)$ is the market price of risk:

$$\sigma_M(\theta, t) = \gamma\sigma_Y - \frac{\frac{1}{\psi} - \gamma}{1 - \gamma}\frac{H_\theta(\theta, t)}{H(\theta, t)}\frac{(\theta_H - \theta)(\theta - \theta_L)}{\sigma_Y}. \tag{8}$$

The aggregate stock market is the claim to a dividend process of the form:

$$\frac{dD_t}{D_t} = \left[\xi\left(\hat{\theta}_t - \bar{\theta}\right) + \bar{\theta}\right] dt + \sigma_Y d\hat{B}_{Y,t}, \tag{9}$$

where ξ is the leverage parameter. The stock price is of the form $p\left(\hat{\theta}_t, t\right) D_t$, where $p\left(\hat{\theta}_t, t\right)$ is the price-to-dividend ratio defined by

$$p\left(\hat{\theta}_t, t\right) = E_t\left[\int_0^\infty \frac{\pi_{t+s}}{\pi_t}\frac{D_{t+s}}{D_t}ds\right]. \tag{10}$$

We provide the expression for the PDE together with the boundary conditions that determines the function $p\left(\hat{\theta}_t, t\right)$ in Appendix 7.2. With the pricing kernel and the price-to-dividend ratio, the market risk premium is given by the following proposition:

Proposition 2. *(Equity premium)*

In the interior of $(nT, (n+1)T)$, the instantaneous risk premium of the asset is given by:

$$E_t\left[\frac{d\left[p\left(\hat{\theta}_t, t\right)D_t\right] + D_t dt}{p\left(\hat{\theta}_t, t\right)}\right] - r\left(\hat{\theta}_t, t\right)dt = \sigma_M\left(\hat{\theta}_t, t\right)\left[\frac{p_\theta\left(\hat{\theta}_t, t\right)}{p\left(\hat{\theta}_t, t\right)}\frac{\left(\theta_H - \hat{\theta}_t\right)\left(\hat{\theta}_t - \theta_L\right)}{\sigma_Y} + \eta\sigma_Y\right]. \tag{11}$$

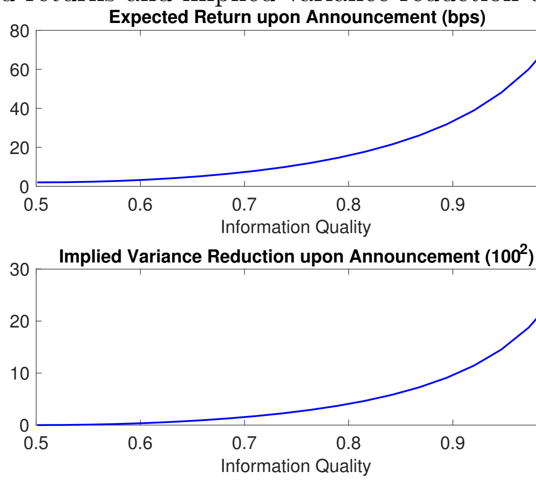
At an announcement time nT, the announcement premium is given by:

$$E_t\left[\frac{p\left(\hat{\theta}_T^+, T^+\right)}{p\left(\hat{\theta}_T^-, T^-\right)}\right] - 1 = \frac{\left(E_T^-\left[H\left(\hat{\theta}_T^+, T^+\right)\right]\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} E_T^-\left[p\left(\hat{\theta}_T^+, T^+\right)\right]}{E_T^-\left[H\left(\hat{\theta}_T^+, T^+\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\hat{\theta}_T^+, T^+\right)\right]} - 1.$$

In our model, $\frac{H\left(\hat{\theta}_T^+, T^+\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}}{E_t\left[H\left(\hat{\theta}_T^+, T^+\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}\right]}$ is the announcement stochastic discount factor. Note that the value function $H\left(\hat{\theta}, t\right)$ is increasing in $\hat{\theta}$. Under the assumption $\gamma > \frac{1}{\psi}$, the term $H\left(\hat{\theta}_T^+, T^+\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}$ is negatively correlated with $p\left(\hat{\theta}_T^+, T^+\right)$. As a result, $Cov\left(H\left(\hat{\theta}_T^+, T^+\right)^{\frac{1}{\psi}-\gamma}, p\left(\hat{\theta}_T^+, T^+\right)\right) < 0$ and the announcement premium is positive.

Comparative statics with respect to informativeness of announcements The key mechanism in our model is that higher informativeness of announcements is associated with a higher expected returns and higher realized variance upon announcements, but lower expected returns and lower realized variance in the post announcement period. This channel induces a negative relationship between past realized variance and future expected returns. In this section, we plot policy functions from our calibrated model to illustrate this basic intuition of the information driven volatility channel.

Figure 4: Expected returns and implied variance reduction upon announcements

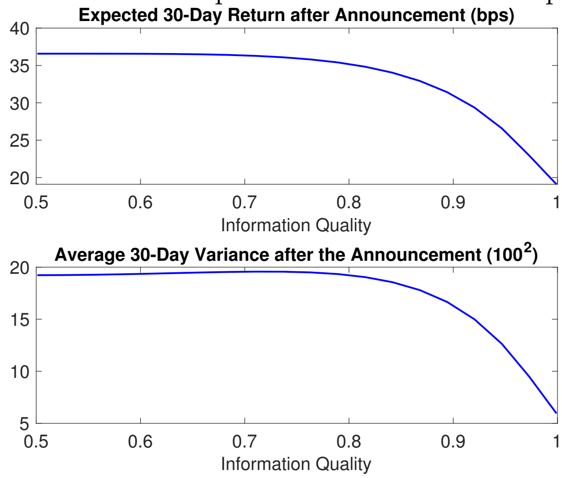


The top panel is the expected announcement-day return as a function of the information quality parameter ν, where expected returns are measured in basis points. The bottom panel is the option implied variance reduction upon announcements implied by our model.

In Figure 4, we plot the expected announcement day return (top panel) and the implied variance reduction (bottom panel) as functions of the information quality parameter ν. Here, implied variance reduction is defined as the difference between option-implied variance right before an announcement at time T, IV_T^-, and the the option implied variance right after the announcement at time T, IV_T^+. Because option implied variance is a forward looking measure of the variance of stock market returns, the difference between the two is a measure of the market expectation of the realized variance on the announcement day T. At $\nu = 0.5$, the announcement is completely uninformative, and the expected return at announcement and the implied variance reduction at announcement are both zero. As ν increases from 0.5 to 1, the expected return and the implied variance reduction upon announcement also rises monotonically. Consistent with the intuition in the three period model, the expected return and the expected variance on announcement days are both increasing functions of the expected informativeness of the announcement.

In Figure 5, we plot the model implied expected return during the 30-day period after the announcement (top panel), and the model implied average realized variance during the same period (bottom panel) as a function of the information quality parameter, ν. Clearly, as the informativeness of announcements increases, the expected returns after announcements reduces and so

Figure 5: Expected returns and implied variance reduction upon announcements



The top panel is the expected 30-day return during the post announcement period as a function of the information quality parameter ν implied by our model. The bottom panel is the average 30-day realized variance during the post announcement period implied by our model.

does the expected realized variance of the market return during the same period. More informative announcements resolve a larger fraction of uncertain about future consumption growth and are associated with lower expected returns and lower realized variance in the post announcement period.

5 Quantitative Results

Parameter values In this section, we calibrate our model and evaluate its implications on the volatility-expected return relationship. We choose a discount rate $\rho = 1.8\%$, a risk aversion $\gamma = 10$, a IES $\psi = 2$ in line with the standard long-run risk literature. We set the volatility of consumption growth $\sigma_Y = 3\%$ to match the volatility of annual consumption growth in the U.S. in our sample period from 1990-2015. We set the value of the two Markov states $\theta_H = 2.0\%$, $\theta_L = -0.8\%$ and the transition probabilities $\lambda_H = 0.08$ and $\lambda_L = 0.26$ as in Ai and Kiku (2013), who estimate these parameters from aggregate consumption data. We choose a leverage parameter $\xi = 2$.

Table 2: Calibrated Parameters

Panel A. Preferences					
ρ	Time discount rate	1.8%			
ψ	IES	2	γ	Relative risk aversion	10
Panel B. Consumption and dividend dynamics					
σ_Y	Endowment growth volatility	3%			
λ_H	θ transition rate (high to low)	0.08	λ_L	θ transition rate (low to high)	0.26
θ_H	High endowment growth state	0.020	θ_L	Low endowment growth state	-0.008
ξ	leverage	2			
Panel C: Information					
ν_H	High ann signal precision state	0.99	ν_L	Low ann signal precision state	0.6
$\frac{1}{T}$	Frequency of announcements	8			

This table displays the calibrated parameter in our model.

The parameters ν_H and ν_L govern the informativeness of the announcements. We set $\nu_H = 0.99$ and $\nu_L = 0.60$ so that our model matches the mean and standard deviation of implied variance reduction on announcement days. Finally, we choose $T = \frac{1}{8}$ so that there are eight announcements per year in our model, matching the frequency of FOMC announcements in the data. All calibrated parameters are listed in Table 2.

Basic statistics of announcement returns and volatility We list the asset pricing moments in the data and in our model in Table 3. Our model produces an average level of the risk-free rate of 1.28% per year, with a standard deviation of 0.48% per year, both moments are fairly close to their data counterparts. The average equity market premium in the model is 6.334% per year, and the standard deviation of market return is 16% per year. Our model produces a significant announcement premium. The average announce-day return is 53 bps and the average non-announcement day return is 1.5 bps. In the data, many other macroeconomic announcements also generate a significant return on announcement days. With only eight announcements per year, our model needs a slightly higher announcement-day return than the average FOMC announcement premium to generate a comparable level of the equity risk premium as in the data.

Table 3: Asset Pricing Moments

Panel A: Aggregate market returns		Data	Model
$E[R_M] - r_f$	Equity premium	7.46%	6.33%
$Std[R_M]$	Vol of market return	18.5%	16%
$E[r_f]$	Average risk-free rate	0.30%	1.28%
$Std[r_f]$	Vol of risk-free rate	1.78%	0.48%
Panel B: Announcement returns		**Data**	**Model**
$E[R_A]$	A-day average return	26 bps	53 bps
$E[R_N]$	NA-day average return	2 bps	1.5 bps
$AC(R_A)$	AC(1) of A-day return	-0.01	-0.12
$E\left[IV_T^- - IV_T^+\right]$	Av. IV reduction	2.1	3.6
$Corr\left(R_A, IV_T^- - IV_T^+\right)$	Corr between A-day return and IV reduction	0.73	0.32
$Corr\left(R_A, RV_{T,T+1}\right)$	Corr between A-day return and RV	0.12	0.10

This table displays the asset pricing moments in the data and implied by the model. The data in Panel A contains the period of 1929.01-2019.12. The data in Panel B contains the period of 1994.09-2019.12.

Our model matches several features of the announcement-day returns in the data. First, the announcement returns are slightly negatively correlated in the data and in the model. In our model, when the previous announcement is more precise, the associated announcement premium is larger. However, this also means that the uncertainty going forward will be lower, and therefore, the premium for the next announcement will be smaller. A negative correlation between announcement returns is another indication of the information-driven volatility channel at work.

Second, the implied variance drops sharply upon announcement. The average reduction of implied variance is 2.14 (monthly bps squared) in the data with a t-statistics of 4.37. The same moment in the model is 3.6 (monthly bps squared). In our model, the implied variance is a forward-looking measure of variance. Because announcements are typically associated with a significant response of the market valuation, the implied variance is high before announcements and low afterwards. The drop in implied variance reflects the informativeness of the announcement. To further illustrate this implied variance reduction associated with information revelation, in Table 4, we report the quantiles of the implied variance reduction in the data and that implied by our model.

Table 4: Implied Variance Reduction

	Q5	Q25	Q50	Q75	Q95
Data	-5.54	-0.26	1.14	3.88	10.00
Model	-9.65	-1.00	2.06	3.54	8.65

This table displays the quantile of implied variance drop on announcement days in the data and that in the model. The data contains the period of 1994.09-2019.12.

It is clear from the variance decomposition formula in Equation (1) that because $Var\left[E\left(C_T|X_t\right)\right] > 0$ implied variance will on average be lower after the announcements. If both C_T and X_t are normally distributed, then $Var\left[C_T|X_t\right]$ will be a constant, and implied variance reduction must always be positive. In general, the variance decomposition only requires that the average drop in implied variance to be positive. In the data, however, the 5th and the 25th percentiles of implied variance drops are both negative. Our model with a two-state Markov chain captures these features of the data as well. The average reduction of implied variance is unambiguously positive. However, there is a significant fraction of observations with increases in the implied variance after announcements.

Testing the information driven volatility channel In this section, we conduct statistical tests for the key implications of the information driven volatility channel, that is, more informative announcements are associated with higher expected return and higher implied variance reduction on announcement days but lower expected return and lower realized variance during the post announcement periods.

To test the predictability of announcement day return by the informativeness of announcements, we first develop a measure of announcement informativeness that uses only information before announcements. Our measure of informativeness of announcements builds on the intuition that more informative announcements are associated with higher implied variance reductions on announcement days.[1] We outline the two steps in the construction of the informativeness measure here and provide the details of the construction in the appendix. In the first step, we use the term structure of option implied volatility to back out the announcement-day implied vari-

[1] Empirically, implied variance reduction on announcement days are highly correlated with the magnitude of announcement day returns. However, we want to be cautious in interpreting this as evidence as the positive relationship between expected announcement-day return and the informativeness of announcements, because it is well known that realized returns are strongly negatively correlated with implied variance. Our contruction of the informativeness measure avoids this type of look-ahead bias in predictability regressions.

ance, AIV. Assume that right before an FOMC announcement, we observe the implied variance for short maturity options, say options with 9 day to maturity, and the implied variance for long maturity options, say options with 30 day to maturity. Assume also that all non-announcement days have the same return variance, but the immediate announcement day has a different return variance. Denote the announcement day return variance as AIV and the non-announcement day implied variance as NIV. The term structure of implied variance therefore allows us to back out the announcement-day implied variance:

$$IV_9 = AIV + 8 \times NIV,$$

$$IV_{30} = AIV + 29 \times NIV.$$

In the second step, we construct our measure of informativeness, denoted $Info$, by normalizing the above constructed AIV by the average realized variance during the past 21 days leading to the announcement:

$$Info_{t-1} = AIV_{t-1} - RV_{t-1},$$

where we use t for the announcement day. The intuition is that if the implied variance for the upcoming announcement day is significantly higher than the realized variance during the pre-announcement period, it must be due to the fact that the market is expecting that the upcoming announcement will be particularly informative.

Our first test of the model is a predictability regression of FOMC announcement-day return and FOMC-day implied variance reduction on the informativeness measure constructed above. We run the following OLS regression:

$$R_t^{FOMC} = \alpha + \beta \times Info_{t-1} + \varepsilon_t, \tag{12}$$

and

$$IV_{t-1} - IV_t = \alpha + \beta \times Info_{t-1} + \varepsilon_t, \tag{13}$$

where R_t^{FOMC} is the announcement-day return, IV_{t-1} is the option implied variance on the day before the announcement, IV_t is the option implied variance on the announcement day, and $Info_{t-1}$

is the informativeness measure we construct using option prices on the day before the announcement at time $t-1$. We report the regression results in Table 5. As shown in the table, in the return predictability regression, the regression coefficient on $Info_{t-1}$ is positive and significant with a t statistic of 4.70. The above regression has a R^2 of 15.44% in the data and 21.12% in our model. In addition, our measure of informativeness also has a strong predictive power on implied variance reduction with a R^2 of 22.20% in the data. These results strongly support the mechanism of information driven volatility especially given that daily returns are notoriously hard to predict in the data.

Table 5: Model implied predictability by informativeness measure

		R_t^{FOMC}	$IV_{t-1} - IV_t$
Data	$Info_{t-1}$	42.93	4.17
		(4.70)	(3.25)
	R^2	15.44%	22.20%
	$Info_{t-1}$	35.26	7.24
Model	R^2	21.2%	72.6%

This table presents the results of the return predictability regressions defined in (12) and the implied variance reduction predictability regression defined in (13).

Our second set of tests are predictability regressions for post-announcement returns and post announcement realized variances by the informativeness of announcement. Because our interest is to predict the returns and realized variances after the announcement, we can use announcement-day drops in implied variance, $IV_{t-1} - IV_t$ directly as the measure of the informativeness of announcement instead of the ex ante measure of $Info_{t-1}$. We consider the following regression specification:

$$RV_{t,t+\Delta} = \alpha + \beta_1 \left[IV_{t-1} - IV_t\right] + \beta_2 RV_{t-2,t-1} + \beta_3 IV_t + \epsilon_{t,t+\Delta}, \tag{14}$$

for the realized variance predictability, and

$$R_{t,t+\Delta} = \alpha + \beta_1 \left[IV_{t-1} - IV_t\right] + \beta_2 RV_{t-2,t-1} + \beta_3 IV_t + \epsilon_{t,t+\Delta}, \tag{15}$$

for the realized return predictability. In the above regressions, $RV_{t,t+\Delta}$ and $R_{t,t+\Delta}$are the realized variance and realized returns, respectively, from the announcement day t (not including the announcement day itself) to Δ days after the announcement day, for various choices of Δ: $\Delta = 1, 2, 3, 4, 5, 21, 42$ up to two months. We include variables that are known to predict realized variances, such as the realized variance on the day before the announcement day, $RV_{t-2,t-1}$, and the implied variance on the announcement day, IV_t, on the right hand side of the regression. Our main interest is the regression coefficient on the measure of the informativeness of the announcement, $IV_{t-1} - IV_t$. We report our regression results in Table 6.

Table 6: Model Implied Return and Variance Predictability by IV Reduction

Number of days		1	2	3	4	5	21	42
$RV_{t,t+\Delta}$	Data	−0.07	−0.04	−0.02	−0.02	−0.02	−0.03	−0.01
		(−4.21)	(−3.52)	(−1.53)	(−1.81)	(−1.93)	(−3.32)	(−0.98)
	Model	−0.25	−0.24	−0.24	−0.23	−0.23	−0.23	−0.23
R^2	Data	0.81	0.76	0.62	0.64	0.67	0.52	0.42
	Model	0.81	0.81	0.81	0.81	0.81	0.81	0.81
$R_{t,t+\Delta}$	Data	−2.00	−2.16	−0.96	−1.21	−0.68	−0.26	−0.23
		(−0.77)	(−1.55)	(−1.42)	(−1.78)	(−1.49)	(−1.62)	(−2.65)
	Model	−1.20	−1.00	−0.90	−0.80	−0.60	−0.06	−0.06
R^2	Data	0.05	0.04	0.03	0.08	0.02	0.02	0.02
	Model	0.12	0.12	0.11	0.11	0.10	0.10	0.10

This table presents the results of the realized variance predictability regression (14) and the return predictability regression (15) in the data and those in our model. The 3-9 columns represent the horizon of returns (in percentage) and variances on the left hand side of Equations (14) and (15), respectively, with $\Delta = 1, 2, 3, 4, 5, 21, 42$ days.

Consistent with the policy functions we plot in Figures 5, in our model, more informative announcements are associated with lower realized variance and lower expected returns after announcements. As a result, the betas in both regressions, (14) and (15), from the model simulated data are negative. Consistent with our model, these regressions show a similar pattern in the data. The drops of implied variance on announcement days negatively predict post-announcement day variances at 1-5 days horizon and this pattern extends to the one-month horizon, but dissipates over time and becomes insignificant over the two-month horizon. This announcement-day drops in implied variance can also negatively predict post-announcement day returns up to two-month horizon. All of the above evidence confirms the basic mechanism of the information driven volatility channel we illustrated in Figures 4 and 5.

Volatility managed portfolios Moreira and Muir (2017) demonstrate that a volatility managed portfolio, that is, a portfolio that takes a high leverage and invests more in the market portfolio after low realized volatility and that reduces leverage after high realized volatility earns a higher return than the market portfolio. Our model provides a simple explanation for this result: because information driven volatility induces a negative relationship between past realized volatility and future expected returns. In this section, we construct the volatility managed portfolio in our model evaluate the ability of our model to quantitatively account for the return on the volatility managed portfolio.

We simulate our model for 1000 months. We follow Moreira and Muir (2017) and compute a cumulative return for each month denoted f_{t+1}, where time is measured in months. For each $t+1$, we use the daily return of the previous month to construct the realized volatility:

$$RV_t^2(f) = \sum_{d=1/30}^{1} \left(f_{t+d} - \frac{1}{30} \sum_{d=1/22}^{1} f_{t+d} \right)^2 .$$

The buy-and-hold strategy is the market return constructed from the sequence of $\{f_{t+1}\}$. The volatility managed portfolio is constructed as

$$f_{t+1}^{\sigma} = \frac{c}{RV_t^2(f)} f_{t+1},$$

where the constant c is chosen so that managed portfolio $\left\{ f_{t+1}^{\sigma} \right\}$ and the market portfolio have the same unconditional standard deviation.

In our simulated model, the average monthly return of the buy-and-hold market portfolio is 0.46% and the average monthly return on the volatility managed portfolio is 0.606% per month. This feature of our model matches quite well the volatility managed portfolio documented by Moreira and Muir (2017). In addition, we run a CAPM regression of the volatility managed portfolio returns on the buy-and-hold market returns. We obtain a CAPM alpha of 3.76% and a beta of 0.70 at the annual level. Both numbers of close to their empirical counterparts in the data, 4.08% and 0.61, respectively.

Variance risk premium predictability In this section, we report our model's implications on VRP return predictability regressions. Previous literature has documented a robust empirical evidence on return predictability by the difference between option implied variance and realized variance up to six-month horizons. The stochastic volatility models that were developed to address this empirical phenomenon typically relied on high-frequency variations of the the volatility of aggregate consumption, for example, Bollerslev, Tauchen, and Zhou (2009) and Drechsler and Yaron (2011). Our model features homoscedastic shocks of macroeconomic fundamentals. However, information driven volatility creates a wedge between implied and realized volatility. The difference between implied and realized variance in our model reflects the informativeness of the upcoming announcement. Because implied variance is a forward-looking measure of variance, it increases when the upcoming announcement is expected to be informative. The difference between implied and realized variance predicts returns because more informative announcements are associated with higher realizations of announcement premiums.

Table 7: Return predictability by $IV - RV$

Number of days		21	42	63	84	105	126
$IV_t - RV_t$	Data	0.03	0.04	0.08	0.13	0.10	0.08
		(2.95)	(1.67)	(2.83)	(5.59)	(2.57)	(1.87)
R^2 (%)		1.62	1.81	4.47	9.39	3.99	2.34
$IV_t - RV_t$	Model	0.03	0.05	0.05	0.05	0.05	0.06

This table presents the results of the return predictability regression (2) using U.S. stock market return data and those using data simulated from the model . Columns 2 to 7 represent returns on the left hand side of (2) with $\Delta = [21, 42, 63, 84, 105, 126]$ trading days. The regression includes returns and VRPs every 21 days during the period of 1990.01.01-2019.12.31. Newey-West t-statistics with 1-6 lags are in parentheses.

In Table 7, we report the results of the return predictability regression (2). As in the data, in our model, returns are predictable by the differences between IV and RV. The regression coefficient of returns on $IV_t - RV_t$ are significant up to the six month horizon in our model, despite the fact that our model does not have a variance risk premium due to homoscedastic fundamental shocks.

Our purpose is not to argue for the absence of the variance risk premium. The level of VIX implied variance in the data is clearly higher than the historical average of realized stock market variance Both the variance premium channel and the information driven volatility channel may contribute the above return predictability evidence in the data. The advantage of our model is that it does not rely on high frequency variations in the volatility of macroeconomic fundamentals.

6 Conclusion

In this paper, we present a model of information-driven volatility. Traditional models of stochastic volatility typically imply a positive relationship between the realized variance of past returns and the forward-looking future returns. However, empirical evidence often favors a negative relationship between the two, which is exemplified by the evidence on volatility managed portfolio returns. We develop a model of information-driven volatility. We show that when variations in stock market volatility are driven by information, high realized variances of past return typically predict lower future variances and lower future returns. We show that our model can account for several stylized facts on the variance-expected return relationships in the data.

It is not our purpose to argue for the absence of stochastic volatility of macroeconomic fundamentals. We do believe that the stochastic volatility of macroeconomic fundamentals are important and will affect the risk-return relationship in the long run. Our purpose is to demonstrate the the information driven volatility channel is more likely to drive the volatility-return relationship over high frequencies, and we present a simple equilibrium model to evaluate the quantitative relevance of this channel. Given the different implications of information driven volatility and fundamental driven volatility on the volatility-expected return relationship, it is important to identify the driving forces of volatility shocks in empirical investigations of the volatility-expected return relationship.

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7 Appendix

7.1 The Two-Period Model

Below, we provide the proof for Proposition 1 of the paper.

Proof. Standard Bayesian updating implies that the posterior mean $E[\ln C_1|s] = \frac{\sigma^{-2}}{\sigma^{-2}+\sigma_\epsilon^{-2}}\mu + \frac{\sigma_\epsilon^{-2}}{\sigma^{-2}+\sigma_\epsilon^{-2}}s$. In addition, $Var[E[\ln C_1|s]] = \lambda\sigma^2$, and $Var[\ln C_1|s] = (1-\lambda)\sigma^2$ where $\lambda = \frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}$, which the variance decomposition formula in (1).

The expressions of the stochastic discount factor is also standard. The stochastic discount factor that prices date-0 consumption goods in terms of date--1 consumption goods is $\Lambda_{-1,0} = \frac{\beta}{1-\beta}\left(\frac{C_0}{C_{-1}}\right)^{-1}\left(\frac{V_0}{m_{-1}}\right)^{1-\gamma}$, and the stochastic discount that prices data-1 consumption goods in terms of date 0 consumption goods is given by: $\Lambda_{0,1} = \frac{\beta}{1-\beta}\left(\frac{C_1}{C_0}\right)^{-1}\left(\frac{C_1}{m_0}\right)^{1-\gamma}$, where $V_0 = C_0^{1-\beta}\left(E\left[C_1^{1-\gamma}\right]\right)^{\frac{\beta}{1-\gamma}}$, is the date-0 utility of the agent, $m_0 = \left(E\left[C_1^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}$ is the certainty equivalent of future at time 0 and $m_{-1} = \left(E\left[V_0^{1-\gamma}\right]\right)^{\frac{1}{1-\gamma}}$ is the date--1 certainty equivalent of future utility.

Therefore, $Var[\ln\Lambda_{0,1}] = \gamma^2 Var[\ln C_1|s] = \gamma^2(1-\lambda)\sigma^2$. Note also, $Var[\ln\Lambda_{-1,0}] = (1-\gamma)^2 Var[\ln V_0]$, where

$$\begin{aligned}\ln V_0 &= (1-\beta)\ln C_0 + \frac{\beta}{1-\gamma}\ln E\left[C_1^{1-\gamma}\right] \\ &= (1-\beta)\ln C_0 + \beta\left\{E[\ln C_1|s] + \frac{1}{2}(1-\gamma)Var[\ln C_1|s]\right\}.\end{aligned}$$

This implies that $Var[\ln\Lambda_{-1,0}] = (1-\gamma)^2\beta^2 Var[E[\ln C_1|s]] = (1-\gamma)^2\beta^2\lambda\sigma^2$. Given the expression for $\lambda = \frac{\sigma^2}{\sigma^2+\sigma_\epsilon^2}$, Proposition 1 can be easily proved. □

7.2 The Infinite Horizon Model

The filtering equations The intensity matrix of the continuous-time Markov chain is:

$$\begin{bmatrix} -\lambda_H & \lambda_H \\ \lambda_L & -\lambda_L \end{bmatrix}.$$

Intuitively, λ_H is the rate of transition from high to low, and λ_L is the rate of transition from low to high. That is, the transition matrix over a small interval Δ is

$$\begin{bmatrix} e^{-\lambda_H \Delta} & 1 - e^{-\lambda_H \Delta} \\ 1 - e^{-\lambda_L \Delta} & e^{-\lambda_L \Delta} \end{bmatrix}.$$

This Markov chain can be conveniently represented as integration with respect to Poisson processes. In particular, let $\{N_{j,t}\}_{t \geq 0}$ be a Poisson process with intensity λ_j, for $j = H,\ L$. Let $I_{\{x\}}$ be the indicator function, that is,

$$I_{\{x\}}(y) = \begin{cases} 1 & if\ y = x \\ 0 & if\ y \neq x \end{cases}$$

Then $\{\theta_t\}$ can be represented as the following compound Poisson process:

$$d\theta_t = (\theta_H - \theta_L)\,\eta\left(\theta_t^-\right)^T dN_t \tag{16}$$

and $\eta(\theta)$, and $N(t)$ are vector notations:

$$\eta(\theta) = \left[-I_{\{\theta_H\}}(\theta),\ I_{\{\theta_L\}}(\theta)\right]^T, \tag{17}$$

$$N(t) = [N_{Ht},\ N_{Lt}]^T. \tag{18}$$

Here we use the convention that $\{\theta_t\}$ is right-continuous with left limits, and use the notation

$$\theta_t^- = \lim_{s \to t, s < t} \theta_s$$

To simplify notation, I will use $I_j(\theta)$ for $I_{\{\theta_j\}}(\theta)$ without causing any confusion. $N_{j,t}$ is the counting processes with intensity λ_j. That is,

$$dN_{j,t} = \begin{cases} 1 & with\ prob\ \lambda_j dt \\ 0 & with\ prob\ 1 - \lambda_j dt \end{cases}.$$

Define $\pi_t = P_t(\theta_t = \theta_H)$, and $\hat{\theta}_t = E_t[\theta_t]$, that is, $\hat{\theta}_t = \pi_t\theta_H + (1-\pi_t)\theta_L$, then

$$d\pi_t = [\lambda_L - (\lambda_H + \lambda_L)\pi_t]\,dt + \pi_t(1-\pi_t)(\theta_H - \theta_L)\frac{1}{\sigma_Y}d\hat{B}_{Y,t}, \tag{19}$$

where $\hat{B}_{Y,t}$ is the innovation process defined by:

$$d\hat{B}_{Y,t} = \frac{1}{\sigma_Y}\left[\frac{dY_t}{Y_t} - \hat{\theta}_t dt\right]. \tag{20}$$

Note that the mapping between $\hat{\theta}$ and π is one-to-one. So we can equivalently use $\hat{\theta}$ as the state variable. By definition, $\hat{\theta}_t = \pi_t\theta_H + (1-\pi_t)\theta_L$. Using Ito's lemma,

$$d\hat{\theta}_t = (\lambda_H + \lambda_L)\left(\bar{\theta} - \hat{\theta}_t\right)dt + \left(\theta_H - \hat{\theta}_t\right)\left(\hat{\theta}_t - \theta_L\right)\frac{1}{\sigma_Y}d\hat{B}_{Y,t}, \tag{21}$$

where $\bar{\theta}$ is the steady-state mean of θ:

$$\bar{\theta} = \frac{\lambda_L\theta_H + \lambda_H\theta_L}{\lambda_L + \lambda_H}. \tag{22}$$

Preference We can write down the value function for recursive preference: $V\left(\hat{\theta}, t, Y\right) = H\left(\hat{\theta}, t\right)Y$. The representative consumer's preference is specified by a pair of aggregators $(f, \mathcal{A})$ such that:

$$dV_t = [-f(Y_t, V_t) - \frac{1}{2}\mathcal{A}(V_t)||\sigma_V(t)||^2]dt + \sigma_V(t)dB_t \tag{23}$$

We adopt the convenient normalization $\mathcal{A}(V) = 0$. Duffie and Epstein, and denote $\bar{f}$ the normalized aggregator. Under this normalization, $\bar{f}(Y, V)$ is:

$$\bar{f}(Y, V) = \frac{\rho}{1 - 1/\psi}\frac{Y^{1-1/\psi} - ((1-\gamma)V)^{\frac{1-1/\psi}{1-\gamma}}}{((1-\gamma)V)^{\frac{1-1/\psi}{1-\gamma}-1}} \tag{24}$$

The HJB for recursive utility is

$$\bar{f}\left(Y_t, V\left(\hat{\theta}, t, Y\right)\right) + \mathcal{L}\left[V\left(\hat{\theta}, t, Y\right)\right] = 0. \tag{25}$$

Consider

$$V\left(\hat{\theta}_t, t, Y\right) = \frac{1}{1-\gamma} H\left(\hat{\theta}_t, t\right) Y_t^{1-\gamma} \tag{26}$$

where

$$\begin{aligned} \frac{dY_t}{Y_t} &= \hat{\theta}_t dt + \sigma_Y d\hat{B}_{Y,t}, & (27) \\ d\hat{\theta}_t &= \mu\left(\hat{\theta}_t\right) dt + \sigma\left(\hat{\theta}_t\right)\left(\frac{1}{\sigma_s} d\hat{B}_{s,t} + \frac{1}{\sigma_Y} d\hat{B}_{Y,t}\right), & (28) \end{aligned}$$

where $\mu(\theta) = (\lambda_H + \lambda_L)\left(\bar{\theta} - \theta\right)$, $\sigma(\theta) = (\theta_H - \theta)(\theta - \theta_L)$. Using generalized Ito's formula, the HJB equation is written as

$$\begin{aligned} 0 = & \frac{1}{(1-\gamma) H}\left\{H_t + H_\theta\left[\mu_{\theta,t} + (1-\gamma)\sigma_{\theta,t}\right] + \frac{1}{2} H_{\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right)\right\} \\ & + \frac{\rho}{1-\frac{1}{\psi}}\left(H^{-\frac{1-\frac{1}{\psi}}{1-\gamma}} - 1\right) + \left(\hat{\theta}_t - \frac{1}{2}\gamma\sigma_Y^2\right), \end{aligned} \tag{29}$$

At the boundary,

$$H\left(\pi_T^-, T\right) = \mathbb{E}\left[H\left(\pi_T^+, 0\right) \mid \pi_T^-, T\right] \tag{30}$$

With this, we can write down the law of motion for the pricing kernel

$$\frac{dM_t}{M_t} = \frac{d\bar{f}_Y(Y,V)}{f_Y(Y,V)} + \bar{f}_V(Y,V)\, dt \tag{31}$$

where $\bar{f}_Y(Y,V) = \rho H^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} Y_t^{-\gamma}$, and $\bar{f}_V(Y,V) = \rho\frac{\frac{1}{\psi}-\gamma}{1-\frac{1}{\psi}} H^{-\frac{1-\frac{1}{\psi}}{1-\gamma}} - \rho\frac{1-\gamma}{1-\frac{1}{\psi}}$. Applying Ito's lemma, we can derive the the pricing kernel in Equation ().

Learning on the boundary At the boundary, given the distribution of θ, π, we need to compute the distribution of $\hat{\theta}_T^+$, or π_T^+. Applying Bayes' rule,

$$P^+(\theta_i|s_j) = \frac{P(s_j|\theta_i)\, P^-(\theta_i)}{\sum_{\theta_i\in\Theta} P(s_j|\theta_i)\, P^-(\theta_i)}.$$

That is, given that $P^-(\theta_H) = \pi^-$, we have:

$$P^+(\theta_H|s_H) = \frac{\pi^-\nu}{\pi^-\nu + (1-\pi^-)(1-\nu)};\ P^+(\theta_L|s_H) = \frac{(1-\pi^-)(1-\nu)}{\pi^-\nu + (1-\pi^-)(1-\nu)} = 1 - P^+(\theta_H|s_H), \tag{32}$$

and

$$P^+(\theta_H|s_L) = \frac{\pi^-(1-\nu)}{\pi^-(1-\nu) + (1-\pi^-)\nu};\ P^+(\theta_L|s_L) = 1 - P^+(\theta_H|s_L). \tag{33}$$

Now, given $P^-(\theta_H) = \pi^-$, we need to compute the distribution of π^+. If we see s_H, then, $\pi^+ = \frac{\pi^-\nu}{\pi^-\nu+(1-\pi^-)(1-\nu)}$, and if we see s_L, $\pi^+ = \frac{\pi^-(1-\nu)}{\pi^-(1-\nu)+(1-\pi^-)\nu}$. So π^+ has only two possible realizations. The probability of seeing s_H is $\pi^-\nu + (1-\pi^-)(1-\nu)$ and the probability of seeing s_L is $\pi^-(1-\nu) + (1-\pi^-)\nu$. Therefore, if we use π as the state variable, the boundary condition for value function (see equation (30)) is

$$\begin{aligned} H\left(\pi_T^-, T\right) &= \mathbb{E}\left[H\left(\pi_T^+, 0\right) \mid \pi_T^-, T\right] \\ &= \left[\pi^-\nu + \left(1-\pi^-\right)(1-\nu)\right] H\left(\frac{\pi^-\nu}{\pi^-\nu + (1-\pi^-)(1-\nu)}, 0\right) \\ &+ \left[\pi^-(1-\nu) + \left(1-\pi^-\right)\nu\right] H\left(\frac{\pi^-(1-\nu)}{\pi^-(1-\nu) + (1-\pi^-)\nu}, 0\right). \\ &= h_{s_H} H\left(\pi_{s_H}^+, 0\right) + h_{s_L} H\left(\pi_{s_L}^+, 0\right) \end{aligned} \tag{34}$$

where $h_{s_H} = [\pi^-\nu + (1-\pi^-)(1-\nu)]$, $h_{s_L} = [\pi^-(1-\nu) + (1-\pi^-)\nu]$, $\pi_{s_H}^+ = \frac{\pi^-\nu}{\pi^-\nu+(1-\pi^-)(1-\nu)}$, and $\pi_{s_L}^+ = \frac{\pi^-(1-\nu)}{\pi^-(1-\nu)+(1-\pi^-)\nu}$.

Note that our signal generates the "correct" result with probability ν and produces a wrong signal with probability $1-\nu$. In simulations, given π^-, and given ν, we need to simulate π^+. If the true state is θ_H, then we set

$$\pi^+ = \begin{cases} \pi_{s_H}^+ = \frac{\pi^-\nu}{\pi^-\nu+(1-\pi^-)(1-\nu)} & \textit{with prob } \nu \\ \pi_{s_L}^+ = \frac{\pi^-(1-\nu)}{\pi^-(1-\nu)+(1-\pi^-)\nu} & \textit{with prob } 1-\nu. \end{cases}$$

If the true state is θ_L, then we set

$$\pi^+ = \begin{cases} \pi^+_{s_L} = \frac{\pi^-(1-\nu)}{\pi^-(1-\nu)+(1-\pi^-)\nu} & \textit{with prob } \nu \\ \pi^+_{s_H} = \frac{\pi^-\nu}{\pi^-\nu+(1-\pi^-)(1-\nu)} & \textit{with prob } 1-\nu. \end{cases}$$

If we want to keep using $\hat{\theta}$ as the state variable, note that $\hat{\theta} = \pi\theta_H + (1-\pi)\theta_L = \theta_L + \pi(\theta_H - \theta_L)$, that is, we can recover π from $\hat{\theta}$: $\pi = \frac{\hat{\theta}-\theta_L}{\theta_H-\theta_L}$. In addition, given π^+, we can compute $\hat{\theta}^+ = \theta_L + \pi^+(\theta_H - \theta_L)$. Therefore,

$$h_{s_H} \equiv \pi^-\nu + \left(1-\pi^-\right)(1-\nu) = \frac{\hat{\theta}-\theta_L}{\theta_H-\theta_L}\nu + \frac{\theta_H-\hat{\theta}}{\theta_H-\theta_L}(1-\nu), \tag{35}$$

$$h_{s_L} \equiv \pi^-(1-\nu) + \left(1-\pi^-\right)\nu = \frac{\hat{\theta}-\theta_L}{\theta_H-\theta_L}(1-\nu) + \frac{\theta_H-\hat{\theta}}{\theta_H-\theta_L}\nu. \tag{36}$$

Also,

$$\hat{\theta}^+_{s_H}\Big|_{\pi^+_{s_H}=\frac{\pi^-\nu}{\pi^-\nu+(1-\pi^-)(1-\nu)}} = \theta_L + \frac{\pi^-\nu}{\pi^-\nu+(1-\pi^-)(1-\nu)}(\theta_H-\theta_L) \tag{37}$$

$$= \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)\nu(\theta_H-\theta_L)}{\left(\hat{\theta}^- - \theta_L\right)\nu + \left(\theta_H - \hat{\theta}^-\right)(1-\nu)}, \tag{38}$$

and

$$\hat{\theta}^+_{s_L}\Big|_{\pi^+_{s_L}=\frac{\pi^-(1-\nu)}{\pi^-(1-\nu)+(1-\pi^-)\nu}} = \theta_L + \frac{\pi^-(1-\nu)}{\pi^-(1-\nu)+(1-\pi^-)\nu}(\theta_H-\theta_L) \tag{39}$$

$$= \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)(1-\nu)(\theta_H-\theta_L)}{\left(\hat{\theta}^- - \theta_L\right)(1-\nu) + \left(\theta_H - \hat{\theta}^-\right)\nu}. \tag{40}$$

Therefore, if the underlying state is $\theta_T = \theta_H$, then we set

$$\hat{\theta}^+_T = \begin{cases} \hat{\theta}^+_{s_H} = \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)\nu(\theta_H-\theta_L)}{\left(\hat{\theta}^- - \theta_L\right)\nu + \left(\theta_H - \hat{\theta}^-\right)(1-\nu)} & \textit{with prob } \nu \\ \hat{\theta}^+_{s_L} = \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)(1-\nu)(\theta_H-\theta_L)}{\left(\hat{\theta}^- - \theta_L\right)(1-\nu) + \left(\theta_H - \hat{\theta}^-\right)\nu} & \textit{with prob } 1-\nu. \end{cases} \tag{41}$$

If the true state is θ_L, then we set

$$\hat{\theta}_T^+ = \begin{cases} \hat{\theta}_{s_L}^+ = \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)(1-\nu)(\theta_H - \theta_L)}{\left(\hat{\theta}^- - \theta_L\right)(1-\nu) + \left(\theta_H - \hat{\theta}^-\right)\nu} & with\ prob\ \nu \\ \hat{\theta}_{s_H}^+ = \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)\nu(\theta_H - \theta_L)}{\left(\hat{\theta}^- - \theta_L\right)\nu + \left(\theta_H - \hat{\theta}^-\right)(1-\nu)} & with\ prob\ 1-\nu. \end{cases} \tag{42}$$

The unconditional distribution of $\hat{\theta}_T^+$ is

$$\hat{\theta}_T^+ = \begin{cases} \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right) v_{T^-} (\theta_H - \theta_L)}{\left(\hat{\theta}^- - \theta_L\right) v_{T^-} + \left(\theta_H - \hat{\theta}^-\right)\left(1 - v_{T^-}\right)} & w.p. \quad \frac{\hat{\theta}^- - \theta_L}{\theta_H - \theta_L} v_{T^-} + \frac{\theta_H - \hat{\theta}^-}{\theta_H - \theta_L}\left(1 - v_{T^-}\right) \\ \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)\left(1 - v_{T^-}\right)(\theta_H - \theta_L)}{\left(\hat{\theta}^- - \theta_L\right)\left(1 - v_{T^-}\right) + \left(\theta_H - \hat{\theta}^-\right) v_{T^-}} & w.p. \quad \frac{\hat{\theta}^- - \theta_L}{\theta_H - \theta_L}\left(1 - v_{T^-}\right) + \frac{\theta_H - \hat{\theta}^-}{\theta_H - \theta_L} v_{T^-} \end{cases}. \tag{43}$$

It is straightforward to verify that if $v_{T^-} = 1$, then the signal is perfectly informative, and the above becomes

$$\hat{\theta}_T^+ = \begin{cases} \theta_H & w.p. \quad \frac{\hat{\theta}^- - \theta_L}{\theta_H - \theta_L} \\ \theta_L & w.p. \quad \frac{\theta_H - \hat{\theta}^-}{\theta_H - \theta_L} \end{cases}.$$

And if $v_{T^-} = 0.5$, then the signal does nothing and the above becomes $\hat{\theta}_T^+ = \hat{\theta}^-$ with probability one.

As a matter of notation, for any function f we denote

$$\begin{aligned} E\left[f\left(\hat{\theta}_T^+\right)|\nu, \hat{\theta}^-\right] = &\left[\frac{\hat{\theta}^- - \theta_L}{\theta_H - \theta_L} v + \frac{\theta_H - \hat{\theta}^-}{\theta_H - \theta_L}(1-v)\right] f\left(\theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right) v\,(\theta_H - \theta_L)}{\left(\hat{\theta}^- - \theta_L\right) v + \left(\theta_H - \hat{\theta}^-\right)(1-v)}\right) \\ &+ \left[\frac{\hat{\theta}^- - \theta_L}{\theta_H - \theta_L}(1-v) + \frac{\theta_H - \hat{\theta}^-}{\theta_H - \theta_L} v\right] f\left(\theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)(1-v)(\theta_H - \theta_L)}{\left(\hat{\theta}^- - \theta_L\right)(1-v) + \left(\theta_H - \hat{\theta}^-\right) v}\right) \end{aligned} \tag{44}$$

For example, we can rewrite

$$H\left(\hat{\theta}^{-},T\right)=h_{s_H}H\left(\hat{\theta}^{+}_{s_H},T\right)+h_{s_L}H\left(\hat{\theta}^{+}_{s_L},T\right) \tag{45}$$

$$=\left[\frac{\hat{\theta}^{-}-\theta_L}{\theta_H-\theta_L}\nu+\frac{\theta_H-\hat{\theta}^{-}}{\theta_H-\theta_L}\left(1-\nu\right)\right]H\left(\theta_L+\frac{\left(\hat{\theta}^{-}-\theta_L\right)\nu\left(\theta_H-\theta_L\right)}{\left(\hat{\theta}^{-}-\theta_L\right)\nu+\left(\theta_H-\hat{\theta}^{-}\right)\left(1-\nu\right)},0\right) \tag{46}$$

$$+\left[\frac{\hat{\theta}^{-}-\theta_L}{\theta_H-\theta_L}\left(1-\nu\right)+\frac{\theta_H-\hat{\theta}^{-}}{\theta_H-\theta_L}\nu\right]H\left(\theta_L+\frac{\left(\hat{\theta}^{-}-\theta_L\right)\left(1-\nu\right)\left(\theta_H-\theta_L\right)}{\left(\hat{\theta}^{-}-\theta_L\right)\left(1-\nu\right)+\left(\theta_H-\hat{\theta}^{-}\right)\nu},0\right). \tag{47}$$

Asset Prices Specify the dividend growth rate as follows

$$\frac{dD_t}{D_t}=\left[\xi\left(\hat{\theta}_t-\bar{\theta}\right)+\bar{\theta}\right]dt+\eta\sigma_Y d\hat{B}_{Y,t}. \tag{48}$$

Now the stock price can be solved as $P\left(\hat{\theta}_t,t,D_t\right)=p\left(\hat{\theta}_t,t\right)D$, where $p\left(\hat{\theta}_t,t\right)$ is the price-to-dividend ratio, which is characterize by the form:

$$M\left(t\right)D_t dt+\mathcal{L}\left[M\left(t\right)p\left(\hat{\theta}_t,t\right)D_t\right]=0. \tag{49}$$

Therefore, the PDE for $p\left(\hat{\theta}_t,t\right)$ is

$$\varpi\left(\hat{\theta}_t,t\right)p_j \quad = \quad p_t+p_\theta\varrho\left(\hat{\theta}_t,t\right)+\frac{1}{2}p_{\theta\theta}\sigma^2_{\theta,t}\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)+1 \tag{50}$$

where

$$\begin{aligned}\varpi\left(\hat{\theta}_t,t\right) &= -\bar{\theta}\left(1-\xi\right)+\rho-\frac{1}{2}\gamma\sigma_Y^2\left(\frac{1}{\psi}+1\right)+\gamma\eta\sigma_Y^2-\left(\xi-\frac{1}{\psi}\right)\hat{\theta}_t-\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\sigma_{\theta,t}\left(\eta-1\right)\frac{H_\theta}{H}\\ &\quad+\frac{\left(\frac{1}{\psi}-\gamma\right)\left(1-\frac{1}{\psi}\right)}{2\left(1-\gamma\right)^2}\left(\frac{H_\theta}{H}\right)^2\sigma^2_{\theta,t}\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)\end{aligned} \tag{51}$$

$$\varrho\left(\hat{\theta}_t,t\right) \quad = \quad \mu_{\theta,t}+\left(\eta-\gamma\right)\sigma_{\theta,t}+\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\frac{H_\theta}{H}\sigma^2_{\theta,t}\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{52}$$

At the boundary,

$$p\left(\pi_T^-, T\right) = \mathbb{E}_T\left[\frac{H\left(\pi_T^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\pi_T^+, 0\right)}{\mathbb{E}_T\left[H\left(\pi_T^+, 0\right) \mid \pi_T^-\right]^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}}\right] \tag{53}$$

$$= \frac{h_{s_H} H\left(\pi_{s_H}^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\pi_{s_H}^+, 0\right) + h_{s_L} H\left(\pi_{s_L}^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\pi_{s_L}^+, 0\right)}{\left[h_{s_H} H\left(\pi_{s_H}^+, 0\right) + h_{s_L} H\left(\pi_{s_L}^+, 0\right)\right]^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}} \tag{54}$$

or in terms of $\hat{\theta}$,

$$p\left(\hat{\theta}_T^-, T\right) = \mathbb{E}_T\left[\frac{H\left(\hat{\theta}_T^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\hat{\theta}_T^+, 0\right)}{\mathbb{E}_T\left[H\left(\hat{\theta}_T^+, 0\right)\right]^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}}\right] \tag{55}$$

$$= \frac{h_{s_H} H\left(\theta_{s_H}^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\theta_{s_H}^+, 0\right) + h_{s_L} H\left(\theta_{s_L}^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\theta_{s_L}^+, 0\right)}{\left[h_{s_H} H\left(\theta_{s_H}^+, 0\right) + h_{s_L} H\left(\theta_{s_L}^+, 0\right)\right]^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}} \tag{56}$$

where $h_{s_H} = \frac{\hat{\theta}-\theta_L}{\theta_H-\theta_L}\nu + \frac{\theta_H-\hat{\theta}}{\theta_H-\theta_L}(1-\nu)$, $h_{s_L} = \frac{\hat{\theta}-\theta_L}{\theta_H-\theta_L}(1-\nu) + \frac{\theta_H-\hat{\theta}}{\theta_H-\theta_L}\nu$. Also, $\hat{\theta}_{s_H}^+ = \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)\nu(\theta_H-\theta_L)}{\left(\hat{\theta}^- - \theta_L\right)\nu + \left(\theta_H - \hat{\theta}^-\right)(1-\nu)}$, and $\hat{\theta}_{s_L}^+ = \theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)(1-\nu)(\theta_H-\theta_L)}{\left(\hat{\theta}^- - \theta_L\right)(1-\nu) + \left(\theta_H - \hat{\theta}^-\right)\nu}$.

Risk Premium Conjecture the cumulated return of the following form:

$$\frac{dR_t}{R_t} = \mu_{R,t}dt + \sigma_{RY,t}d\hat{B}_{Y,t} + \sigma_{Rs,t}d\hat{B}_{s,t} \tag{57}$$

The cumulative return can be computed as

$$\frac{dR_t}{R_t} = \frac{1}{p\left(\hat{\theta}_t, t\right)D_t}\left[D_t dt + d\left(p\left(\hat{\theta}_t, t\right)D_t\right)\right] = \frac{1}{p}dt + \frac{d\left(pD_t\right)}{pD_t} \tag{58}$$

$$\frac{d\left(p\left(\hat{\theta}_t, t\right)D_t\right)}{p\left(\hat{\theta}_t, t\right)D_t} = \left\{\frac{1}{p}\left[p_t + p_\theta\mu_{\theta,t} + \frac{1}{2}p_{\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right)\right] + \xi\left(\hat{\theta}_t - \bar{\theta}\right) + \bar{\theta} + \frac{p_\theta}{p}\eta\sigma_{\theta,t}\right\}dt$$

$$+ \left(\frac{p_\theta}{p}\frac{\sigma_{\theta,t}}{\sigma_Y} + \eta\sigma_Y\right)d\hat{B}_{Y,t} + \frac{p_\theta}{p}\frac{\sigma_{\theta,t}}{\sigma_s}d\hat{B}_{s,t} \tag{59}$$

Therefore

$$\mu_{R,t} = \frac{1}{p}\left[1+p_t+p_\theta\mu_{\theta,t}+\frac{1}{2}p_{\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)\right]+\xi\left(\hat{\theta}_t-\bar{\theta}\right)+\bar{\theta}+\frac{p_\theta}{p}\eta\sigma_{\theta,t} \tag{60}$$

$$\sigma_{RY,t} = \frac{p_\theta}{p}\frac{\sigma_{\theta,t}}{\sigma_Y}+\eta\sigma_Y \tag{61}$$

$$\sigma_{Rs,t} = \frac{p_\theta}{p}\frac{\sigma_{\theta,t}}{\sigma_s} \tag{62}$$

Together with the expression of pricing kernel, the risk premium is therefore

$$\frac{dM_t}{M_t} = -r_t dt-\sigma_{MY,t}d\hat{B}_{Y,t}-\sigma_{Ms,t}d\hat{B}_{s,t} \tag{63}$$

$$\sigma_{MY,t} = \gamma\sigma_Y-\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\frac{H_\theta}{H}\frac{\sigma_{\theta,t}}{\sigma_Y}. \tag{64}$$

$$\sigma_{Ms,t} = -\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\frac{H_\theta}{H}\frac{\sigma_{\theta,t}}{\sigma_s}. \tag{65}$$

$$\mathbb{E}_t\left[\frac{dR_t}{R_t}\right]-r_t = -\mathrm{Cov}_t\left[\frac{dM_t}{M_t},\frac{dR_t}{R_t}\right] \tag{66}$$

$$\mu_{R,t}-r_t = \left(\gamma\sigma_Y-\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\frac{H_\theta}{H}\frac{\sigma_{\theta,t}}{\sigma_Y}\right)\left(\frac{p_\theta}{p}\frac{\sigma_{\theta,t}}{\sigma_Y}+\eta\sigma_Y\right)-\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\frac{H_\theta}{H}\left(\frac{\sigma_{\theta,t}}{\sigma_s}\right)^2\frac{p_\theta}{p} \tag{67}$$

Computing returns On non-announcement days:

$$R_{t,t+\Delta} = \frac{p\left(\hat{\theta}_{t+\Delta},t+\Delta\right)D_{t+\Delta}+\int_t^{t+\Delta}D_s ds}{p\left(\hat{\theta}_t,t\right)D_t}.$$

Note that $D_{t+\Delta} = D_t e^{\int_t^{t+\Delta}\left(\xi\hat{\theta}_s+\bar{\theta}(1-\xi)-\frac{1}{2}\eta^2\sigma_Y^2\right)ds+\int_t^{t+\Delta}\eta\sigma_Y d\hat{B}_{Y,s}}$. As an approximation,

$$\frac{D_{t+\Delta}}{D_t} = e^{\xi\frac{1}{2}\left[\hat{\theta}_t+\hat{\theta}_{t+\Delta}\right]\Delta+\left[\bar{\theta}(1-\xi)-\frac{1}{2}\eta^2\sigma_Y^2\right]\Delta+\eta\sigma_Y\left(\hat{B}_{Y,t+\Delta}-\hat{B}_Y\right)}$$

and

$$\frac{\int_t^{t+\Delta}D_s ds}{D_t} = \frac{1}{2}\Delta\frac{D_{t+\Delta}+D_t}{D_t} = \frac{1}{2}\Delta\left(1+e^{\xi\frac{1}{2}\left[\hat{\theta}_t+\hat{\theta}_{t+\Delta}\right]\Delta+\left[\bar{\theta}(1-\xi)-\frac{1}{2}\eta^2\sigma_Y^2\right]\Delta+\eta\sigma_Y\left(\hat{B}_{Y,t+\Delta}-\hat{B}_Y\right)}\right)$$

On announcement day, assume that the announcement happens at the end of the day. First calculate non-announcement return $R_{T-\Delta,T}$ as above. Note calculate

$$R_A = \frac{p\left(\nu_T, \hat{\theta}_T^+, T^+\right)}{p\left(\nu_T, \hat{\theta}_T^-, T^-\right)}, \tag{68}$$

where $\hat{\theta}_T^+$ is drawn from the distribution described in (43). The total return equals $R_{T-\Delta,T} \times R_A$.

7.3 Forward Looking Measures of Variance

7.3.1 Expressing variance as expectations

To compute implied variance, we need to compute

$$Var_t\left[\ln\left\{p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) D_\tau\right\} - \ln\left\{p\left(\nu_t, \hat{\theta}_t, t\right) D_t\right\}\right] = Var_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) + \ln D_\tau\right]. \tag{69}$$

Note that $D_\tau = D_t e^{\int_t^\tau \left(\xi\hat{\theta}_s + \bar{\theta}(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2\right)ds + \int_t^\tau \eta\sigma_Y d\hat{B}_{Y,s}}$ and therefore $\ln D_\tau = \ln D_t + \int_t^\tau (\xi\hat{\theta}_s + \bar{\theta}\,(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2)ds + \int_t^\tau \eta\sigma_Y d\hat{B}_{Y,s}$. Therefore,

$$\begin{aligned}
&Var_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) + \ln D_\tau\right] \\
&= Var_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) + \int_t^\tau \left(\xi\hat{\theta}_s + \bar{\theta}\,(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2\right) ds + \int_t^\tau \eta\sigma_Y d\hat{B}_{Y,s}\right]. \\
&= Var_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\right] + 2Cov_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right), \delta(\tau) - \delta(t)\right] + Var_t\left[\delta(\tau) - \delta(t)\right],
\end{aligned} \tag{70}$$

where we denote

$$\delta(t) = \int_0^t \left(\xi\hat{\theta}_s + \bar{\theta}\,(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2\right) ds + \int_0^t \eta\sigma_Y d\hat{B}_{Y,s}, \tag{71}$$

or equivalently,

$$d\delta(t) = \left[\xi\hat{\theta}_t + \bar{\theta}\,(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2\right] dt + \eta\sigma_Y d\hat{B}_{Y,t}. \tag{72}$$

Let's compute the above step by step. We first consider the log return from t to τ:

$$\begin{aligned}&\ln\left\{p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)D_\tau\right\}-\ln\left\{p\left(\nu_t,\hat{\theta}_t,t\right)D_t\right\}\\ =&\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)-\ln p\left(\nu_t,\hat{\theta}_t,t\right)+\int_t^\tau(\xi\hat{\theta}_s+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2)ds+\int_t^\tau\eta\sigma_Y d\hat{B}_{Y,s}.\end{aligned} \tag{73}$$

Then the log return in (73) can be written as:

$$\ln\left\{p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)D_\tau\right\}-\ln\left\{p\left(\nu_t,\hat{\theta}_t,t\right)D_t\right\}=\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)-\ln p\left(\nu_t,\hat{\theta}_t,t\right)+\delta\left(\tau\right)-\delta\left(t\right). \tag{74}$$

To compute the variance of the log return, we have:

$$\begin{aligned}&Var_t\left[\ln\left\{p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)D_\tau\right\}-\ln\left\{p\left(\nu_t,\hat{\theta}_t,t\right)D_t\right\}\right]\\ =&Var_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]+2Cov_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right),\delta\left(\tau\right)-\delta\left(t\right)\right]+Var_t\left[\delta\left(\tau\right)-\delta\left(t\right)\right],\end{aligned} \tag{75}$$

where we use the fact that $\ln p\left(\nu_t,\hat{\theta}_t,t\right)$ is known at time t. Note that the system is defined by two Markov state variables, $\left(\hat{\theta},\nu\right)$, but $\delta\left(t\right)$ is NOT one of the state variables. This gives rise to some complications in the computation of expectations, which we have to deal with separately.

The three terms in Equation (75) can be written as:

$$Var_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]=E_t\left[\ln^2 p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]-\left(E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]\right)^2, \tag{76}$$

$$Cov_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right),\delta\left(\tau\right)-\delta\left(t\right)\right]=E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\delta\left(\tau\right)\right]-E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]E_t\left[\delta\left(\tau\right)\right], \tag{77}$$

and

$$Var_t\left[\delta\left(\tau\right)-\delta\left(t\right)\right]=E_t\left[\delta^2\left(\tau\right)\right]-\left(E_t\left[\delta\left(\tau\right)\right]\right)^2. \tag{78}$$

Decomposition

1. The functions $\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)$ and $\ln^2 p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)$ depends only on the value of the Markov state variables $\left(\nu,\hat{\theta},t\right)$ at time τ. Therefore, the terms $E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]$ and $E_t\left[\ln^2 p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]$ can be computed using Lemma 2. Note if there is an announcement in between t and τ, then

the PDE is not enough and we need the three step procedure.

2. The terms $\delta^2(\tau)$, $\delta(\tau)$, and $\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\delta(\tau)$ depends on the entire path of the BM, and cannot be computed using Lemma 2. We will need to derive the PDE for the expectations $E_t\left[\delta^2(\tau)\right]$, $E_t[\delta(\tau)]$, and $E_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\delta(\tau)\right]$ separately. Below we first state a sequence of Lemmas to compute the integrals.

3. In order to use the martingale method to derive an expression for the above expressions, it is convenient to define a few integrals related to $\delta(t)$ as follows:

$$a_1\left(\nu, \hat{\theta}, t\right) = E_t\left[\delta(\tau) - \delta(t)\right], \tag{79}$$

$$a_0\left(\nu, \hat{\theta}, t\right) = E_t\left[\{\delta(\tau) - \delta(t)\}^2\right], \tag{80}$$

and

$$a_3\left(\nu, \hat{\theta}, t\right) = E_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\{\delta(\tau) - \delta(t)\}\right]. \tag{81}$$

With the above definition of $a_0\left(\nu, \hat{\theta}, t\right)$, $a_1\left(\nu, \hat{\theta}, t\right)$, and $a_3\left(\nu, \hat{\theta}, t\right)$, it is not hard to show that the implied variance can constructed as follows:

Lemma 1. *The implied variance is given by:*

$$\begin{aligned} Var_t\left[lnp\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) + lnD_\tau\right] = {} & E_t\left[\ln^2 p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\right] - \left(w\left(\nu_t, \hat{\theta}_t, t\right)\right)^2 \\ & + 2\left[a_3\left(\nu_t, \hat{\theta}_t, t\right) - w\left(\nu_t, \hat{\theta}_t, t\right) a_1\left(\nu_t, \hat{\theta}_t, t\right)\right] \\ & + a_0\left(\nu_t, \hat{\theta}_t, t\right) - a_1\left(\nu_t, \hat{\theta}_t, t\right)^2, \end{aligned} \tag{82}$$

where

$$w\left(\nu_t, \hat{\theta}_t, t\right) = E_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\right]. \tag{83}$$

Proof. Note that three lines in (82) correspond to the three equations (76)-(78). The first line is obvious. The second line is:

$$\begin{aligned}
&E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\delta(\tau)\right]-E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]E_t\left[\delta(\tau)\right]\\
=&E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\{\delta(\tau)-\delta(t)\}\right]-E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]E_t\left[\delta(\tau)-\delta(t)\right]\\
=&a_3\left(\nu_t,\hat{\theta}_t,t\right)-w\left(\nu_t,\hat{\theta}_t,t\right)a_1\left(\nu_t,\hat{\theta}_t,t\right).
\end{aligned}$$

Similarly, to compute equation (78), we need $E_t\left[\delta^2(\tau)\right]$ and $\left(E_t\left[\delta(\tau)\right]\right)^2$. We have

$$\begin{aligned}
E_t\left[\delta^2(\tau)\right]&=\delta_t^2+2\delta_t E_t\left[\delta(\tau)-\delta(t)\right]+E_t\left[\{\delta(\tau)-\delta(t)\}^2\right].\\
&=\delta_t^2+2a_1\left(\nu_t,\hat{\theta}_t,t\right)\delta_t+a_0\left(\nu_t,\hat{\theta}_t,t\right).
\end{aligned}$$

Therefore, to compute (78), we have:

$$\begin{aligned}
E_t\left[\delta^2(\tau)\right]-\left(E_t\left[\delta(\tau)\right]\right)^2&=\delta_t^2+2a_1\left(\nu_t,\hat{\theta}_t,t\right)\delta_t+a_0\left(\nu_t,\hat{\theta}_t,t\right)-\left[\delta(t)+a_1\left(\nu_t,\hat{\theta}_t,t\right)\right]^2\\
&=a_0\left(\nu_t,\hat{\theta}_t,t\right)-a_1\left(\nu_t,\hat{\theta}_t,t\right)^2, \qquad (84)
\end{aligned}$$

which completes our proof. □

7.3.2 Computing integrals

Computing $w\left(\nu_t,\hat{\theta}_t,t|\tau\right)$ We first state a lemma that compute the time-t expectation of a payoff delivered at time $\tau>t$.

Lemma 2. *Let $\Psi\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)$ be a function of the Markov state variables, define*

$$w\left(\nu_t,\hat{\theta}_t,t\right)=E_t\left[\Psi\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right].$$

Suppose $0<t<\tau<T$. Then $w\left(\nu_t,\hat{\theta}_t,t\right)$ is determined by the PDE $\mathcal{L}w\left(\nu_t,\hat{\theta}_t,t\right)=0$ and the boundary condition $w\left(\nu,\hat{\theta},\tau\right)=\Psi\left(\nu,\hat{\theta},\tau\right)$ for all $\left(\nu,\hat{\theta}\right)$.

COMMENT: Note for simplicity, we have written $w\left(\nu_t,\hat{\theta}_t,t\right)$ as a function of $w\left(\nu_t,\hat{\theta}_t,t\right)$. Perhaps a better way to write this is $w\left(\nu_t,\hat{\theta}_t,t|\tau\right)$ to emphasize that $w\left(\nu_t,\hat{\theta}_t,t|\tau\right)$ also depends on τ. Numerically, we think about the computation where given τ, we use finite difference to compute $\left\{w\left(\nu_t,\hat{\theta}_t,t|\tau\right)\right\}_{t\in(0,\tau)}$.

The above lemma computes expectations of the form $E_t\left[\Psi\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]$ only for $\tau<T$. If $\tau>T$, which is often the case if we want to talk about implied variance across an announcement, we need to deal with the *announcement boundary* separately, and we do it in three steps.

1. In the first step, we use the above lemma to compute

$$w\left(\nu_{T^+},\hat{\theta}_{T^+},T^+\right)=E_{T^+}\left[\Psi\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]. \tag{85}$$

   That is, for all values of $\left(\nu,\hat{\theta}\right)$, we need to compute

$$w\left(\nu,\hat{\theta},T^+\right)=E\left[\Psi\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)|\nu_{T^+}=\nu,\hat{\theta}_{T^+}=\hat{\theta},t=T^+\right]$$

2. In the next step, we compute the boundary condition:

$$w\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)=E_{T^-}\left[w\left(\nu_{T^+},\hat{\theta}_{T^+},T^+\right)\right]. \tag{86}$$

   That is, for all values of $\left(\nu,\hat{\theta}\right)$, we need to compute $w\left(\nu,\hat{\theta},T^-\right)=E_{T^-}\left[w\left(\nu,\hat{\theta}_{T^+},T^+\right)|\hat{\theta}_{T^-}=\hat{\theta}\right]$. This is a simple calculation that only uses the distribution of $\hat{\theta}_{T^+}$ given $\hat{\theta}_{T^-}$ (see 37) and (39).

3. In the final step, we use the lemma to compute

$$w\left(\nu_t,\hat{\theta}_t,t\right)=E_t\left[\Psi\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)\right].$$

Remark 1. The integrals $E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]$ and $E_t\left[\ln^2 p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]$ in Equations (79) and (80) can be computed using Lemma 2.

Computing $a_1\left(\nu_t,\hat{\theta}_t,t|\tau\right)$ Again, for a fixed τ, our purpose is to compute $\left\{a_1\left(\nu_t,\hat{\theta}_t,t|\tau\right)\right\}_{t\in[0,\tau]}$. For simplicity, we will suppress τ whenever it does not cause any confusion. First, we derive a PDE (together with a boundary condition at $t=\tau$) that characterize $a_1\left(\nu_t,\hat{\theta}_t,t\right)$. By definition,

$$a_1\left(\nu_t,\hat{\theta}_t,t\right)+\delta(t)=E_t\left[\delta(\tau)\right],$$

which means that in the interior, we must have $\mathcal{L}\left[a_1\left(\nu_t,\hat{\theta}_t,t\right)+\delta\left(t\right)\right]=0$. This give a PDE of the form:

$$\mathcal{L}\left[a_1\left(\nu_t,\hat{\theta}_t,t\right)\right]+\left[\xi\hat{\theta}_t+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right]=0, \tag{87}$$

together with the boundary condition $a_1\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)=0$.

Again, we need to deal with the announcement boundary separately. At T:

$$a_1\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)+\delta\left(T^-\right)=E_{T^-}\left[a_1\left(\nu_{T^+},\hat{\theta}_{T^+},T^+\right)+\delta\left(T^+\right)\right].$$

Because $\delta\left(t\right)$ is continuous, $\delta\left(T^-\right)=E_{T^-}\left[\delta\left(T^+\right)\right]=\delta\left(T^+\right)$. $a_1\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)$ must also satisfy the tower property. That is,

$$a_1\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)=E_{T^-}\left[a_1\left(\nu_{T^+},\hat{\theta}_{T^+},T^+\right)\right]. \tag{88}$$

We summarize the above results using the following lemma.

Lemma 3. *Let $a_1\left(\nu,\hat{\theta},t|\tau\right)$ be define as in Equation (79). Suppose $0<t<\tau<T$, then $a_1\left(\nu,\hat{\theta},t\right)$ can be solved by the PDE (87) together with the boundary condition $a_1\left(\nu,\hat{\theta},\tau\right)=0$ for all $\left(\nu,\hat{\theta}\right)$.*

If $\tau>T$, we first compute $a_1\left(\nu_{T^+},\hat{\theta}_{T^+},T^+\right)$ using the above PDE. In the second step, we use (88) to compute $a_1\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)$ at the announcement boundary. With the boundary condition, we apply the PDE again to solve for the entire path of the $a_1\left(\nu,\hat{\theta},t|\tau\right)$ function.

Here, we note that $a_1\left(\nu_t,\hat{\theta}_t,t\right)$ has a closed form solution. Starting from the definition:

$$a_1\left(\hat{\theta}_t,t\right)=E_t\left[\int_t^\tau\left(\xi\hat{\theta}_s+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right)ds+\int_t^\tau\eta\sigma_Y d\hat{B}_{Y,s}\right],$$

$$a_1\left(\hat{\theta}_t,t\right)=\xi\left[\int_t^\tau E_t\left(\hat{\theta}_s\right)ds\right]+\left[\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right]\left(\tau-t\right).$$

Recall that

$$\begin{aligned}\hat{\theta}_\tau&=\bar{\theta}+\left(\theta_t-\bar{\theta}\right)e^{-(\lambda_H+\lambda_L)(\tau-t)}\\&\quad+e^{-(\lambda_H+\lambda_L)(\tau-t)}\int_0^{\tau-t}e^{(\lambda_H+\lambda_L)u}\left(\theta_H-\hat{\theta}_{t+u}\right)\left(\hat{\theta}_{t+u}-\theta_L\right)\left(\frac{1}{\sigma_s}d\hat{B}_{s,t+u}+\frac{1}{\sigma_Y}d\hat{B}_{Y,t+u}\right)\end{aligned} \tag{89}$$

Clearly, for $s > t$,

$$E_t\left(\hat{\theta}_s\right) = \bar{\theta} + \left(\hat{\theta}_t - \bar{\theta}\right) e^{-(\lambda_H+\lambda_L)(s-t)},$$

and

$$\left[\int_t^\tau E_t\left(\hat{\theta}_s\right) ds\right] = (\tau - t)\,\bar{\theta} + \left(\hat{\theta}_t - \bar{\theta}\right) \frac{1}{\lambda_H + \lambda_L}\left[1 - e^{-(\lambda_H+\lambda_L)(\tau-t)}\right]$$

Therefore,

$$\begin{aligned} a_1\left(\hat{\theta}_t, t\right) &= \xi\left\{(\tau - t)\,\bar{\theta} + \left(\hat{\theta}_t - \bar{\theta}\right)\frac{1}{\lambda_H + \lambda_L}\left[1 - e^{-(\lambda_H+\lambda_L)(\tau-t)}\right]\right\} + \left[\bar{\theta}\,(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2\right](\tau - t) \\ &= \left[\bar{\theta} - \frac{1}{2}\eta^2\sigma_Y^2\right](\tau - t) + \frac{\xi}{\lambda_H + \lambda_L}\left[1 - e^{-(\lambda_H+\lambda_L)(\tau-t)}\right]\left(\hat{\theta}_t - \bar{\theta}\right). \end{aligned}$$

Note that the above calculation applies only to the interior case because (89) applies to only the interior and does not incorporate the possibility of an announcement boundary. However, because $a_1\left(\hat{\theta}_t, t\right)$ is linear, having an announcement boundary will actually not affect the above calculation.

Computing $a_0\left(\nu_t, \hat{\theta}_t, t|\tau\right)$ Next, consider $a_0\left(\nu_t, \hat{\theta}_t, t\right)$. Note that we can write $E_t\left[\delta^2(\tau)\right]$ as:

$$E_t\left[\delta^2(\tau)\right] = \delta_t^2 + 2a_1\left(\nu_t, \hat{\theta}_t, t\right)\delta_t + a_0\left(\nu_t, \hat{\theta}_t, t\right). \tag{90}$$

The fact that $E_t\left[\delta^2(\tau)\right]$ is a MG implies that in the interior, the follow condition must satisfy:

$$\mathcal{L}\left[\delta_t^2 + 2a_1\left(\nu_t, \hat{\theta}_t, t\right)\delta_t + a_0\left(\nu_t, \hat{\theta}_t, t\right)\right] = 0 \tag{91}$$

together with the boundary condition $a_0\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) = 0$ has to hold.

At the boundary,

$$E_{T^-}\left[\delta_{T+}^2 + 2a_1\left(\hat{\theta}_T^+, T^+\right)\delta_{T+} + a_0\left(\hat{\theta}_T^+, T^+\right)\right] = \delta_T^2 + 2E_{T^-}\left[a_1\left(\hat{\theta}_T^+, T^+\right)\right]\delta_T + E_{T-}\left[a_0\left(\hat{\theta}_T^+, T^+\right)\right].$$

Again, because $\delta(t)$ is continuous, we must have

$$a_0\left(\nu, \hat{\theta}_T^-, T^-\right) = E_{T-}\left[a_0\left(\hat{\theta}_T^+, T^+\right)\right].$$

These conditions fully characterize $a_0\left(\nu_t,\hat{\theta}_t,t\right)$. We summarize the above results in the following lemma.

Lemma 4. *Let* $a_0\left(\nu,\hat{\theta},t|\tau\right)$ *be define as in Equation (80). Suppose* $0<t<\tau<T$*, then* $a_0\left(\nu,\hat{\theta},t\right)$ *can be solved by the PDE (91) together with the boundary condition* $a_0\left(\nu,\hat{\theta},\tau\right)=0$ *for all* $\left(\nu,\hat{\theta}\right)$*.*

If $\tau>T$*, we first compute* $a_0\left(\nu_{T^+},\hat{\theta}_{T^+},T^+\right)$ *using the above PDE. In the second step, we use (88) to compute* $a_0\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)$ *at the announcement boundary. With the boundary condition, we apply the PDE again to solve for the entire path of the entire* $a_0\left(\nu,\hat{\theta},t|\tau\right)$ *function.*

Computing $a_3\left(\nu_t,\hat{\theta}_t,t|\tau\right)$ We start by summarizing our result in the following lemma.

Lemma 5. *Let* $a_3\left(\nu,\hat{\theta},t|\tau\right)$ *be define as in Equation (81). Suppose* $0<t<\tau<T$*, then* $a_3\left(\nu,\hat{\theta},t\right)$ *can be solved by the following PDE:*

$$\mathcal{L}\left[w\left(\nu_t,\hat{\theta}_t,t\right)\delta\left(t\right)+a_3\left(\nu_t,\hat{\theta}_t,t\right)\right]=0, \tag{92}$$

together with the boundary condition $a_3\left(\nu,\hat{\theta},\tau\right)=0$ *for all* $\left(\nu,\hat{\theta}\right)$*.*

If $\tau>T$*, we first compute* $a_3\left(\nu_{T^+},\hat{\theta}_{T^+},T^+\right)$ *using the above PDE. In the second step, we compute* $a_0\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)$ *at the announcement boundary by using*

$$a_3\left(\nu_{T^-},\hat{\theta}_{T^-},T^-\right)=E_{T^-}\left[a_3\left(\nu_T^+,\hat{\theta}_T^+,T^+\right)\right]. \tag{93}$$

With the boundary condition, we apply the PDE again to solve for the entire path of the entire $a_3\left(\nu,\hat{\theta},t|\tau\right)$ *function.*

Proof. By definition, $a_3\left(\nu_t,\hat{\theta}_t,t\right)=E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\{\delta\left(\tau\right)-\delta\left(t\right)\}\right]$. Therefore,

$$a_3\left(\nu_t,\hat{\theta}_t,t\right)+E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\delta\left(t\right)\right]$$

is a MG and must satisfy Equation (92) together with the boundary condition:

$$a_3\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)=0. \tag{94}$$

At the boundary T, the MG property implies

$$E_{T^-}\left[a_3\left(\nu_T^+,\hat{\theta}_T^+,T^+\right)+w\left(\nu_T^+,\hat{\theta}_T^+,T^+\right)\delta\left(T^+\right)\right]=a_3\left(\nu_T^-,\hat{\theta}_T^-,T^-\right)+w\left(\nu_T^-,\hat{\theta}_T^-,T^-\right)\delta\left(T^-\right).$$

Because $\delta\left(T^-\right)=\delta\left(T^+\right)$ and $w\left(\nu_T^-,\hat{\theta}_T^-,T^-\right)=E_{T^-}\left[w\left(\nu_T^+,\hat{\theta}_T^+,T^+\right)\right]$, we obtain the boundary condition (93). □

7.3.3 Details of PDE and Boundary Conditions

In this section, we provide the details of the functional forms of PDE. We need to take expectations in two steps. First, we note that the law of motion of our MC state variable is

$$d\hat{\theta}_t=\left(\lambda_H+\lambda_L\right)\left(\bar{\theta}-\hat{\theta}_t\right)dt+\left(\theta_H-\hat{\theta}_t\right)\left(\hat{\theta}_t-\theta_L\right)\left(\frac{1}{\sigma_s}d\hat{B}_{s,t}+\frac{1}{\sigma_Y}d\hat{B}_{Y,t}\right).$$

Given the law of motion of $\hat{\theta}$,the $\delta\left(t\right)$ process is given by (72).

PDE for the w and w_2 First, The PDE for the MG condition $\mathcal{L}w\left(\nu_t,\hat{\theta}_t,t\right)=0$ can be written as follows:

$$\mathcal{L}w\left(\nu_t,\hat{\theta}_t,t\right) = w_t+w_\theta\mu_{\theta,t}+\frac{1}{2}w_{\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)=0 \tag{95}$$

The boundary condition for $E_t\left[\ln p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]$ is, for all $\left(\nu,\hat{\theta}\right)$,

$$w\left(\nu,\hat{\theta},\tau\right)=\ln p\left(\nu,\hat{\theta},\tau\right) \tag{96}$$

The boundary condition for $E_t\left[\ln^2 p\left(\nu_\tau,\hat{\theta}_\tau,\tau\right)\right]$ is, for all $\left(\nu,\hat{\theta}\right)$,

$$w_2\left(\nu,\hat{\theta},\tau\right)=\ln^2 p\left(\nu,\hat{\theta},\tau\right). \tag{97}$$

PDE for a_0 and a_1. First, using (87), and PDE for a_1 can be written as:

$$
\begin{aligned}
da_1\left(\hat{\theta}_t,t\right) &= \left[\frac{\partial}{\partial t}a_1\left(\hat{\theta}_t,t\right)+\frac{\partial}{\partial \theta}a_1\left(\hat{\theta}_t,t\right)\mu_{\theta,t}+\frac{1}{2}\frac{\partial^2}{\partial\theta^2}a_1\left(\hat{\theta}_t,t\right)\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)\right]dt \\
&\quad +\frac{\partial}{\partial\theta}a_1\left(\hat{\theta}_t,t\right)\sigma_{\theta,t}\left(\frac{1}{\sigma_s}d\hat{B}_{s,t}+\frac{1}{\sigma_Y}d\hat{B}_{Y,t}\right) \qquad (98)
\end{aligned}
$$

Therefore, using (91), we can write the PDE for a_0 as:

$$
\begin{aligned}
&2\delta\left(\xi\hat{\theta}_t+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right)+\eta^2\sigma_Y^2+2\delta\left(a_{1,t}+a_{1,\theta}\mu_{\theta,t}+\frac{1}{2}a_{1,\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)\right) \\
&+2a_1\left(\xi\hat{\theta}_t+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right)+2a_{1,\theta}\eta\sigma_{\theta,t}+\left(a_{0,t}+a_{0,\theta}\mu_{\theta,t}+\frac{1}{2}a_{0,\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)\right) = 0 \qquad (99)
\end{aligned}
$$

Because the above equation must hold for all δ_t, the coefficients on δ_t must both be zero.

$$
0 = \xi\hat{\theta}_t+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2+\frac{\partial}{\partial t}a_1\left(\hat{\theta}_t,t\right)+\frac{\partial}{\partial\theta}a_1\left(\hat{\theta}_t,t\right)\mu_{\theta,t}+\frac{1}{2}\frac{\partial^2}{\partial\theta^2}a_1\left(\hat{\theta}_t,t\right)\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \qquad (100)
$$

$$
\begin{aligned}
0 &= 2a_1\left(\hat{\theta}_t,t\right)\left(\xi\hat{\theta}_t+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right)+2\frac{\partial}{\partial\theta}a_1\left(\hat{\theta}_t,t\right)\eta\sigma_{\theta,t}+\eta^2\sigma_Y^2 \\
&\quad +\frac{\partial}{\partial t}a_0\left(\hat{\theta}_t,t\right)+\frac{\partial}{\partial\theta}a_0\left(\hat{\theta}_t,t\right)\mu_{\theta,t}+\frac{1}{2}\frac{\partial^2}{\partial\theta^2}a_0\left(\hat{\theta}_t,t\right)\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \qquad (101)
\end{aligned}
$$

with the boundary condition

$$
a_1\left(\hat{\theta}_\tau,\tau\right)=0 \text{ and } a_0\left(\hat{\theta}_\tau,\tau\right)=0. \qquad (102)
$$

The PDE for a_3. Now, we write the PDE associated with the MG condition (92):

$$
\begin{aligned}
0 &= \delta\left(w_t+w_\theta\mu_{\theta,t}+\frac{1}{2}w_{\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)\right)+w\left(\xi\hat{\theta}_t+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right) \\
&\quad +w_\theta\eta\sigma_{\theta,t}+\frac{\partial}{\partial t}a_3\left(\nu_j,\hat{\theta}_t,t\right)+\frac{\partial}{\partial\theta}a_3\left(\nu_j,\hat{\theta}_t,t\right)\mu_{\theta,t}+\frac{1}{2}\frac{\partial^2}{\partial\theta^2}a_3\left(\nu_j,\hat{\theta}_t,t\right)\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)
\end{aligned}
$$

implies the following PDE

$$
\begin{aligned}
0 &= w\left(\xi\hat{\theta}_t+\bar{\theta}\left(1-\xi\right)-\frac{1}{2}\eta^2\sigma_Y^2\right)+w_\theta\eta\sigma_{\theta,t}+\frac{\partial}{\partial t}a_3\left(\nu_j,\hat{\theta}_t,t\right)+\frac{\partial}{\partial\theta}a_3\left(\nu_j,\hat{\theta}_t,t\right)\mu_{\theta,t} \\
&\quad +\frac{1}{2}\frac{\partial^2}{\partial\theta^2}a_3\left(\nu_j,\hat{\theta}_t,t\right)\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \qquad (103)
\end{aligned}
$$

with boundary condition (94).

Implied volatility summary Using lemma 1, for $t \neq T$, implied variance is

$$Var_t\left[\ln p\left(\hat{\theta}_\tau,\tau\right)+\ln D_\tau\right]=w_2\left(\hat{\theta}_t,t\right)-\left(w_1\left(\hat{\theta}_t,t\right)\right)^2$$
$$+2\left[a_3\left(\hat{\theta}_t,t\right)-w_1\left(\hat{\theta}_t,t\right)a_1\left(\hat{\theta}_t,t\right)\right]+a_0\left(\hat{\theta}_t,t\right)-a_1\left(\hat{\theta}_t,t\right)^2.$$

The above formula warrants some explanation. Note that if $t < T$, then all terms of the above equation are solved using the PDE in the interior. If $t < T$, we need to deal with the announcement boundary separately for all involved functions, as stated in the lemmas above.

7.4 Announcement Return Predictability

Online Appendix

Numerical Solutions

Solve for H function HJB equation:

$$\begin{aligned}0 &= H_t+H_\theta\left[\mu_{\theta,t}+(1-\gamma)\,\sigma_{\theta,t}\right]+\frac{1}{2}H_{\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)+\frac{\rho\,(1-\gamma)}{1-\frac{1}{\psi}}\left(H^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}-H\right)\\ &\quad+(1-\gamma)\left(\hat{\theta}_t-\frac{1}{2}\gamma\sigma_Y^2\right)H \qquad (104)\end{aligned}$$

$$\begin{aligned}(1-\gamma)\left(\frac{\rho}{1-\frac{1}{\psi}}-\hat{\theta}_t+\frac{1}{2}\gamma\sigma_Y^2\right)H &= H_t+H_\theta\left[\mu_{\theta,t}+(1-\gamma)\,\sigma_{\theta,t}\right]+\frac{1}{2}H_{\theta\theta}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)\\ &\quad+\frac{\rho\,(1-\gamma)}{1-\frac{1}{\psi}}H^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} \qquad (105)\end{aligned}$$

Use finite difference method and approximate the functions $H\left(\hat{\theta}_t,t\right)$ at I discrete points in the space dimensions, $\hat{\theta}_i$, $i=1,2,...,I$. Denote $H_i^n=H\left(\hat{\theta}_i,t^n\right)$, where time dimension $n=0,1,2,...,N$.

Denote

$$\beta_i = (1-\gamma)\left(\frac{\rho}{1-\frac{1}{\psi}} - \hat{\theta}_i + \frac{1}{2}\gamma\sigma_Y^2\right), \tag{106}$$

$$u_i^{n+1} = \frac{\rho(1-\gamma)}{1-\frac{1}{\psi}}\left(H_i^{n+1}\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}. \tag{107}$$

Use implicit method to update the value function,

$$\begin{aligned}\beta_i H_i^n &= \frac{H_i^{n+1}-H_i^n}{\Delta t} + u_i^{n+1} + \frac{1}{2}\partial_{\theta\theta}H_i^n\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \\ &+\partial_{\theta,F}H_i^n\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^+ + \partial_{\theta,B}H_i^n\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^-. \end{aligned} \tag{108}$$

Use upwind scheme to approximate the derivatives $\partial_\theta H_i^n$ and $\partial_{\theta\theta}H_i^n$,

$$\begin{aligned}\beta_i H_i^n &= \frac{H_i^{n+1}-H_i^n}{\Delta t} + u_i^{n+1} + \frac{1}{2}\frac{H_{i+1}^n - 2H_i^n + H_{i-1}^n}{\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \\ &+\frac{H_{i+1}^n-H_i^n}{\Delta\hat{\theta}}\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^+ + \frac{H_i^n-H_{i-1}^n}{\Delta\hat{\theta}}\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^-. \end{aligned} \tag{109}$$

Collecting terms and rewrite HJB equation,

$$\beta_i H_i^n = \frac{H_i^{n+1}-H_i^n}{\Delta t} + u_i^{n+1} + H_{i-1}^n x_i + H_i^n(y_i-\kappa_j) + H_{i+1}^n z_i \tag{110}$$

where

$$x_i = -\frac{\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^-}{\Delta\hat{\theta}} + \frac{1}{2\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{111}$$

$$y_i = -\frac{\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^+}{\Delta\hat{\theta}} + \frac{\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^-}{\Delta\hat{\theta}} - \frac{1}{\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{112}$$

$$z_i = \frac{\left[\mu_{\theta,i}+(1-\gamma)\sigma_{\theta,i}\right]^+}{\Delta\hat{\theta}} + \frac{1}{2\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{113}$$

Rewrite in the matrix notation,

$$\beta H^n = u^{n+1} + \mathbf{A}^{n+1}H^n + \frac{H^{n+1}-H^n}{\Delta t}, \tag{114}$$

where $\mathbf{A}^{n+1} = \begin{bmatrix} y_1 & z_1 & 0 & \cdots & 0 \\ x_2 & y_2 & z_2 & 0 & \vdots \\ 0 & x_3 & y_3 & z_3 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots \\ 0 & \cdots & 0 & x_I & y_I \end{bmatrix}$, and

$$H^n = H_i^n = \begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ \vdots \\ H_I \end{bmatrix}, \ \beta = \begin{bmatrix} \beta_1 & 0 & & & \cdots \\ 0 & \beta_2 & & & \\ & & \beta_3 & & \\ & & & \ddots & 0 \\ \vdots & & & & \beta_I \end{bmatrix}, \ u^{n+1} = u_{i,j}^{n+1} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_I \end{bmatrix}.$$

The system can be written as

$$\mathbf{B}^{n+1} H^n = b^{n+1}, \ \mathbf{B}^{n+1} = \left(\frac{1}{\Delta t} + \beta\right) - \mathbf{A}^{n+1}, \ b^{n+1} = u^{n+1} + \frac{1}{\Delta t} H^{n+1}. \tag{115}$$

The boundary condition is

$$H\left(\nu_T, \hat{\theta}^-, T\right) = \mathbb{E}\left[H\left(\nu_T, \theta_T^+, 0\right) \mid \nu_T, \theta_T^-, T\right] \tag{116}$$

$$= \left[\frac{\hat{\theta}^- - \theta_L}{\theta_H - \theta_L}\nu + \frac{\theta_H - \hat{\theta}^-}{\theta_H - \theta_L}(1-\nu)\right] H\left(\theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)\nu\left(\theta_H - \theta_L\right)}{\left(\hat{\theta}^- - \theta_L\right)\nu + \left(\theta_H - \hat{\theta}^-\right)(1-\nu)}\right) \tag{117}$$

$$+ \left[\frac{\hat{\theta}^- - \theta_L}{\theta_H - \theta_L}(1-\nu) + \frac{\theta_H - \hat{\theta}^-}{\theta_H - \theta_L}\nu\right] H\left(\theta_L + \frac{\left(\hat{\theta}^- - \theta_L\right)(1-\nu)\left(\theta_H - \theta_L\right)}{\left(\hat{\theta}^- - \theta_L\right)(1-\nu) + \left(\theta_H - \hat{\theta}^-\right)\nu}\right) \tag{118}$$

Solve for price-to-dividend ratio PDE for $p\left(\hat{\theta}_t, t\right)$:

$$\varpi\left(\hat{\theta}_t, t\right) p_j = p_t + p_\theta \varrho\left(\hat{\theta}_t, t\right) + \frac{1}{2} p_{\theta\theta} \sigma_{\theta,t}^2 \left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) + 1 \tag{119}$$

where

$$\varpi\left(\hat{\theta}_t,t\right) = -\bar{\theta}\left(1-\xi\right)+\rho-\frac{1}{2}\gamma\sigma_Y^2\left(\frac{1}{\psi}+1\right)+\gamma\eta\sigma_Y^2-\left(\xi-\frac{1}{\psi}\right)\hat{\theta}_t-\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\sigma_{\theta,t}\left(\eta-1\right)\frac{H_\theta}{H}$$
$$+\frac{\left(\frac{1}{\psi}-\gamma\right)\left(1-\frac{1}{\psi}\right)}{2\left(1-\gamma\right)^2}\left(\frac{H_\theta}{H}\right)^2\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{120}$$

$$\varrho\left(\hat{\theta}_t,t\right) = \mu_{\theta,t}+\left(\eta-\gamma\right)\sigma_{\theta,t}+\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\frac{H_\theta}{H}\sigma_{\theta,t}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{121}$$

Use finite difference method and approximate the functions $p\left(\hat{\theta}_t,t\right)$ at I discrete points in the space dimensions, $\hat{\theta}_i$, $i=1,2,...,I$. Denote $p_i^n=p\left(\hat{\theta}_i,t^n\right)$, where time dimension $n=0,1,2,...,N$. Denote

$$\varpi_i^{n+1} = -\bar{\theta}\left(1-\xi\right)+\rho-\frac{1}{2}\gamma\sigma_Y^2\left(\frac{1}{\psi}+1\right)+\gamma\eta\sigma_Y^2-\left(\xi-\frac{1}{\psi}\right)\hat{\theta}_i-\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\sigma_{\theta,i}\left(\eta-1\right)\frac{H_{\theta,i}^{n+1}}{H_i^{n+1}}$$
$$+\frac{\left(\frac{1}{\psi}-\gamma\right)\left(1-\frac{1}{\psi}\right)}{2\left(1-\gamma\right)^2}\left(\frac{H_{\theta,i}^{n+1}}{H_i^{n+1}}\right)^2\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{122}$$

$$\varrho_i^{n+1} = \mu_{\theta,i}+\left(\eta-\gamma\right)\sigma_{\theta,i}+\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\frac{H_{\theta,i}^{n+1}}{H_i^{n+1}}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right) \tag{123}$$

Since $-\frac{\frac{1}{\psi}-\gamma}{1-\gamma}\sigma_{\theta,i}\left(\eta-1\right)<0$, $\frac{\left(\frac{1}{\psi}-\gamma\right)\left(1-\frac{1}{\psi}\right)}{2(1-\gamma)^2}<0$, and $\frac{\frac{1}{\psi}-\gamma}{1-\gamma}>0$, so that we approximate $H_{\theta,i}^{n+1}$ using backward method for ϖ_i^{n+1} and forward method for ϱ_i^{n+1}. Backward: $H_{\theta,i}^{n+1}=\frac{H_i^{n+1}-H_{i-1}^{n+1}}{\Delta\hat{\theta}}$ gives $\frac{H_{\theta,i}^{n+1}}{H_i^{n+1}}=\frac{1-H_{i-1}^{n+1}/H_i^{n+1}}{\Delta\hat{\theta}}$. Forward: $H_{\theta,i}^{n+1}=\frac{H_{i+1}^{n+1}-H_i^{n+1}}{\Delta\hat{\theta}}$ gives $\frac{H_{\theta,i}^{n+1}}{H_i^{n+1}}=\frac{H_{i+1}^{n+1}/H_i^{n+1}-1}{\Delta\hat{\theta}}$.

Use implicit method to update the value function,

$$\varpi_i^{n+1}p_i^n = \frac{p_i^{n+1}-p_i^n}{\Delta t}+1+\frac{1}{2}\partial_{\theta\theta}p_i^n\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)$$
$$+\partial_{\theta,F}p_i^n\left(\varrho_i^{n+1}\right)^+ +\partial_{\theta,B}p_i^n\left(\varrho_i^{n+1}\right)^- . \tag{124}$$

Use upwind scheme to approximate the derivatives $\partial_\theta p_i^n$ and $\partial_{\theta\theta}p_i^n$,

$$\varpi_i^{n+1}p_i^n = \frac{p_i^{n+1}-p_i^n}{\Delta t}+1+\frac{1}{2}\frac{p_{i+1}^n-2p_i^n+p_{i-1}^n}{\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2}+\frac{1}{\sigma_Y^2}\right)$$
$$+\frac{p_{i+1}^n-p_i^n}{\Delta\hat{\theta}}\left(\varrho_i^{n+1}\right)^+ +\frac{p_i^n-p_{i-1}^n}{\Delta\hat{\theta}}\left(\varrho_i^{n+1}\right)^- . \tag{125}$$

Collecting terms and rewrite the PDE,

$$\varpi_i^{n+1} p_i^n = \frac{p_i^{n+1} - p_i^n}{\Delta t} + 1 + p_{i-1}^n x_i^{n+1} + p_i^n y_i^{n+1} + p_{i+1}^n z_i^{n+1} \tag{126}$$

where

$$x_i^{n+1} = -\frac{\left(\varrho_i^{n+1}\right)^-}{\Delta\hat{\theta}} + \frac{1}{2\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) \tag{127}$$

$$y_i^{n+1} = -\frac{\left(\varrho_i^{n+1}\right)^+}{\Delta\hat{\theta}} + \frac{\left(\varrho_i^{n+1}\right)^-}{\Delta\hat{\theta}} - \frac{1}{\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) \tag{128}$$

$$z_i^{n+1} = \frac{\left(\varrho_i^{n+1}\right)^+}{\Delta\hat{\theta}} + \frac{1}{2\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) \tag{129}$$

Rewrite in the matrix notation,

$$\varpi^{n+1} p^n = 1 + \mathbf{A}^{n+1} p^n + \frac{p^{n+1} - p^n}{\Delta t}, \tag{130}$$

The system can be written as

$$\mathbf{B}^{n+1} p^n = b^{n+1},\ \mathbf{B}^{n+1} = \left(\frac{1}{\Delta t} + \varpi^{n+1}\right) - \mathbf{A}^{n+1},\ b^{n+1} = 1 + \frac{1}{\Delta t} p^{n+1}. \tag{131}$$

At the boundary,

$$p\left(\nu_T, \hat{\theta}_T^-, T\right) = \frac{h_{s_H} H\left(\nu_T, \theta_{s_H}^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\nu_T, \theta_{s_H}^+, 0\right) + h_{s_L} H\left(\nu_T, \theta_{s_L}^+, 0\right)^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}} p\left(\nu_T, \theta_{s_L}^+, 0\right)}{\left[h_{s_H} H\left(\nu_T, \theta_{s_H}^+, 0\right) + h_{s_L} H\left(\nu_T, \theta_{s_L}^+, 0\right)\right]^{\frac{\frac{1}{\psi}-\gamma}{1-\gamma}}} \tag{132}$$

where

$$h_{s_H} = \pi^- \nu + \left(1 - \pi^-\right)(1 - \nu) = \frac{\hat{\theta} - \theta_L}{\theta_H - \theta_L}\nu + \frac{\theta_H - \hat{\theta}}{\theta_H - \theta_L}(1 - \nu),$$

$$h_{s_L} = \pi^-(1 - \nu) + \left(1 - \pi^-\right)\nu = \frac{\hat{\theta} - \theta_L}{\theta_H - \theta_L}(1 - \nu) + \frac{\theta_H - \hat{\theta}}{\theta_H - \theta_L}\nu.$$

and,

$$\begin{aligned}
\hat{\theta}^{+}_{s_H} &= \theta_L + \frac{\left(\hat{\theta}^{-} - \theta_L\right)\nu\left(\theta_H - \theta_L\right)}{\left(\hat{\theta}^{-} - \theta_L\right)\nu + \left(\theta_H - \hat{\theta}^{-}\right)(1-\nu)}, \\
\hat{\theta}^{+}_{s_L} &= \theta_L + \frac{\left(\hat{\theta}^{-} - \theta_L\right)(1-\nu)\left(\theta_H - \theta_L\right)}{\left(\hat{\theta}^{-} - \theta_L\right)(1-\nu) + \left(\theta_H - \hat{\theta}^{-}\right)\nu}.
\end{aligned}$$

Implied Variance PDEs The PDEs are in general as the form of

$$\begin{aligned}
0 &= \frac{w_i^{n+1} - w_i^n}{\Delta t} + u_i^{n+1} + \frac{1}{2}\frac{w_{i+1}^n - 2w_i^n + w_{i-1}^n}{\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) \\
&\quad + \frac{w_{i+1}^n - w_i^n}{\Delta\hat{\theta}}\mu_{\theta,i}^{+} + \frac{w_i^n - w_{i-1}^n}{\Delta\hat{\theta}}\mu_{\theta,i}^{-}. \qquad (133)
\end{aligned}$$

where $u_i^{n+1} = 0$ for $\hat{w} = E_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\right]$ and $E_t\left[\ln^2 p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\right]$, whereas for a_1, a_0 and a_3,

$$u_i^{n+1} = \xi\hat{\theta}_t + \bar{\theta}(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2 \qquad (134)$$

$$u_i^{n+1} = 2a_{1,i}^{n+1}\left(\xi\hat{\theta}_i + \bar{\theta}(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2\right) + 2\frac{a_{1,i+1}^{n+1} - a_{1,i}^{n+1}}{\Delta\hat{\theta}}\eta\sigma_{\theta,t} + \eta^2\sigma_Y^2 \qquad (135)$$

$$u_i^{n+1} = \hat{w}_i^{n+1}\left(\xi\hat{\theta}_t + \bar{\theta}(1-\xi) - \frac{1}{2}\eta^2\sigma_Y^2\right) + \frac{\hat{w}_{i+1}^{n+1} - \hat{w}_i^{n+1}}{\Delta\hat{\theta}}\eta\sigma_{\theta,t} \qquad (136)$$

Therefore, rewrite

$$0 = \frac{w_i^{n+1} - w_i^n}{\Delta t} + u_i^{n+1} + w_{i-1}^n x_i + w_i^n y_i + w_{i+1}^n z_i \qquad (137)$$

where

$$x_i = -\frac{\mu_{\theta,i}^{-}}{\Delta\hat{\theta}} + \frac{1}{2\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) \qquad (138)$$

$$y_i = -\frac{\mu_{\theta,i}^{+}}{\Delta\hat{\theta}} + \frac{\mu_{\theta,i}^{-}}{\Delta\hat{\theta}} - \frac{1}{\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) \qquad (139)$$

$$z_i = \frac{\mu_{\theta,i}^{+}}{\Delta\hat{\theta}} + \frac{1}{2\left(\Delta\hat{\theta}\right)^2}\sigma_{\theta,i}^2\left(\frac{1}{\sigma_s^2} + \frac{1}{\sigma_Y^2}\right) \qquad (140)$$

Rewrite in the matrix notation,

$$0 = u^{n+1} + \mathbf{A}^{n+1} w^n + \frac{w^{n+1} - w^n}{\Delta t}, \tag{141}$$

The system can be written as

$$\mathbf{B}^{n+1} w^n = b^{n+1},\ \mathbf{B}^{n+1} = \frac{1}{\Delta t}\mathbf{I} - \mathbf{A}^{n+1},\ b^{n+1} = u^{n+1} + \frac{1}{\Delta t} w^{n+1}. \tag{142}$$

with the boundary condition for $E_t\left[\ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\right]$ is:

$$w\left(\nu, \hat{\theta}, \tau\right) = \ln p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) \tag{143}$$

and boundary condition for $E_t\left[\ln^2 p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right)\right]$ is:

$$w\left(\nu, \hat{\theta}, \tau\right) = \ln^2 p\left(\nu_\tau, \hat{\theta}_\tau, \tau\right) \tag{144}$$

and left boundary conditions

$$a_1\left(\hat{\theta}_\tau, \tau\right) = 0,\ a_0\left(\hat{\theta}_\tau, \tau\right) = 0 \text{ and } a_3\left(\nu, \hat{\theta}_\tau, \tau\right) = 0. \tag{145}$$