Lecture 1: No arbitrage pricing

I The fundamental theorem of finance

- $(\Omega, \mathcal{F}, \pi)$, where $\Omega = \{1, 2, \dots, n\}$, and $\pi(\omega) > 0$ for $\omega = 1, \dots, n$.
- J securities $j \in \{1, \dots, J\}$. Let x_j be the payoff of security j, $x_j : \Omega \to \mathbb{R}$
- The prices of the J securities are given by $[p_1, p_2, \cdots, p_J]$. The payoff of the J securities is represented by the following matrix (the payoff matrix):

$$X = \begin{bmatrix} x_1(1) & \dots & x_1(n) \\ \vdots & \vdots & \vdots \\ x_J(1) & \dots & x_J(n) \end{bmatrix}$$

- A portfolio holding strategy is $\varphi = [\varphi_1, \dots, \varphi_J]^T \in \Phi \ \subseteq \mathbb{R}^J$
- Asset Span $\mathcal{M} \in \mathbb{R}^n$ is the set of all payoffs that can be achieved by some portfolio strategy:

$$\mathcal{M} = \{\sum_{j=1}^{J} \varphi_j x_j : \varphi \in \Phi\}$$

  Typically, we take $\Phi = R^J$ when talking about asset spans.

- Complete Market: $\mathcal{M} = \mathbb{R}^n$.
- **Claim**: A sufficient condition for market being complete is that X has full rank.

  Proof. $\forall y = \{y(1), \dots, y(n)\}^T \in \mathbb{R}^n$, need to find $\{\varphi(j)\}_{j=1}^J$, such that $\sum_{j=1}^J \varphi_j x_j = y$, which is equivalent to

$$X^T \varphi = y$$

  Since X is full rank, then we can always find solution to the above equation, hence we know that $\mathbb{R}^n \subseteq \mathcal{M}$, combine with the condition that $\mathcal{M} \subseteq \mathbb{R}^n$, we can get the $\mathcal{M} = \mathbb{R}^n$, which is the definition of complete market. □

- **Definition**: An arbitrage is a portfolio strategy $\varphi \in \Phi$ such that $\sum_{j=1}^J \varphi_j x_j(\omega) \geq 0$, for all $\omega = 1, 2, , n$, and $\sum_{j=1} \varphi_j p_j \leq 0$ with strictly inequality hold at least for one of them.

Theorem 1. *(Fundamental Theorem) The following two statements are equivalent.*

1. *$\nexists$ an arbitrage in $\Phi = \mathbb{R}^J$.*

2. $\exists\, q \in \mathbb{R}^n, q >> 0$ *such that for all* $j = 1, 2, \cdots, J$,

$$p_j = \sum_{\omega=1}^{n} q(\omega) x_j(\omega)$$

for $j = 1, \ldots, J$. *(m is called stochastic discount factor)*

Proof. Farkas" lemma; Stiemke's lemma; Theorem of alternatives. □

Remark 1. *(Implication of the theorem)*

- It is often convenient to define $m(\omega) = \frac{q(\omega)}{\pi(\omega)}$ (as the Radon-Nikodym derivative of q with respect to π) and write:

$$\begin{aligned} p_j &= \sum_{\omega=1}^{n} q(\omega) x_j(\omega) \\ &= \sum_{\omega=1}^{n} \pi(\omega) \left(\frac{q(\omega)}{\pi(\omega)} \right) x_j(\omega) \\ &= E[m x_j] \end{aligned}$$

  In the above representation, m is typically the stochastic discount factor, or simply SDF.

- If market is complete, then m is unique (proof?)
- Proj $m|\mathcal{M}$ is unique. The Proj $m|\mathcal{M}$ is defined as $y = \alpha_0 + \sum_{j=1}^{J} \alpha_j x_j$ such that $E[\{m - y\} x_j] = 0$ for all j.

II Risk premium and factor pricing

A Risk premium

The fundamental theorem of finance implies that for any return R,

$$1 = \mathrm{E}(mR)$$

In particular, in the special case of the risk-free return,

$$R_f = \frac{1}{\mathrm{E}(m)}. \tag{1}$$

We can write the above equation as:

$$\begin{aligned}\mathrm{E}(m)\mathrm{E}(R) + \mathrm{Cov}\,(m, R) &= 1 \\ \mathrm{E}(R) + \frac{\mathrm{Cov}(m, R)}{\mathrm{E}(m)} &= R_f\end{aligned}$$

Therefore,

$$\begin{aligned}\mathrm{E}(R) - R_f &= -\frac{\mathrm{Cov}(m, R)}{\mathrm{E}(m)} \\ &= -\frac{Cov(m, R)}{\mathrm{Var}(m)} \frac{Var(m)}{\mathrm{E}(m)} \end{aligned} \tag{2}$$

The term $\dfrac{Cov(m, R)}{\mathrm{Var}(m)}$ is the regression coefficient of R on the SDF m, of β w.r.t. the SDF. $\dfrac{Var(m)}{\mathrm{E}(m)}$ is the market price of risk.

B CAPM

The SDF is not observable. Empirically, a large volume of literature uses the CAPM. That is,

$$\mathrm{E}(R) - R_f = \beta \left[\mathrm{E}(R^M) - R_f\right],$$

where R^M stands for the market return.

Question: When does CAPM hold? Give examples of GE models where the CAPM holds (approximately).

C Factor Pricing

Theorem 2. *Suppose we have k factors $\{f_1, \ldots, f_K\}$ where $f_k : \Omega \to \mathbb{R}$ for all k. Assume that $m \in \mathcal{M}\{f_1, \ldots, f_K\}$. That is, the SDF is spanned by the set of factors, i.e., there exists coefficients $\{\alpha_k\}_{k=0}^K$ such that $m = \alpha_0 + \sum_{k=1}^K \alpha_k f_k$. Then the risk premium for any return R satisfies:*

$$\mathrm{E}(R) - R_f = \sum_{k=1}^{K} \beta_k \gamma_k \tag{3}$$

where $\{\beta_k\}_{k=1}^K$ is the coefficients of a multi-variate regression of R on the set of factors, and γ_k is the market price of risk for factor k defined by:

$$\gamma_k = -\frac{Cov\,(m, f_k)}{E\,[m]}. \tag{4}$$

In particular, if $\{f_1, \ldots, f_K\}$ is a vector of returns, then $\{\gamma_k\}$ is the risk premium of the returns.

Proof. For any return R, consider the projection of the return R onto the set of factors $\{f_1, \ldots, f_k\}$:

$$R = \alpha(0) + \sum_{k=1}^{K} \beta_k f_k + \varepsilon. \tag{5}$$

By construction, we know $\varepsilon \perp F$, where F is the linear space spanned by the factors. Because $m \in F$, we have $E[m\varepsilon] = 0$.

Equation (5) therefore implies:

$$\mathrm{E}[mR_j] = \alpha_0 \mathrm{E}[m] + \sum_{k=1}^{K} \beta_k \mathrm{E}(mf_k) + \mathrm{E}(m\varepsilon)$$

That is,

$$1 = \frac{\alpha_0}{R_f} + \sum_{k=1}^{K} \beta_k \mathrm{E}(mf_k),$$

or

$$R_f = \alpha_0 + \sum_{k=1}^{K} \beta_k \frac{\mathrm{E}[mf_k]}{E[m]}. \tag{6}$$

Taking expectation on both sides of (5) and subtract (6), we have:

$$\begin{aligned} E[R] - R_f &= \sum_{k=1}^{K} \beta_k \left\{ E[f_k] - \frac{\mathrm{E}[mf_k]}{E[m]} \right\} \\ &= -\sum_{k=1}^{K} \beta_k \left\{ \frac{Cov(m, f_k)}{E[m]} \right\}, \end{aligned}$$

which is (3). If $\{f_k\}$ are returns, then by (2), $\{\gamma_k\}$ are risk premiums. □

D The Hansen-Jaganathan Bound

Equation (2) implies

$$\mathrm{E}(R) - R_f = -\frac{\mathrm{Cov}(m, R)}{\mathrm{E}(m)} = -\rho \frac{\sigma(m)\,\sigma(R)}{E[m]} \leq \frac{\sigma(m)\,\sigma(R)}{E[m]}$$

Therefore,

$$\sigma(m) \geq \frac{1}{R_f} \frac{\mathrm{E}(R) - R_f}{\sigma(R)}.$$

The std of the SDF is bounded from below by the maximum Sharpe ratio of any asset.

III Application

A Consumption based asset pricing

Suppose an investor's utility can be represented by $u\left(c_0\right) + \beta E\left[u\left(C_1\right)\right]$. Consider the investor's optimal portfolio choice problem:

$$\max_{\{c_0, \{c_1(s)\}, \{\varphi_j\}\}} \left\{ u(c_0) + \beta \sum_{s=1}^{n} \pi(s) u(c_1(s)) \right\}$$

$$c_0 + \sum_{j=1}^{J} \varphi_j p_j = w$$

$$c_1(s) = \sum_{j=1}^{J} \varphi_j x_j(s)$$

First order condition:

$$u'(c_0) p_j - \beta \sum_{s=1}^{n} \pi(s) u'(c_1(s)) x_j(s) = 0$$

which is equivalent to

$$\mathrm{E}\left(\frac{\beta u'(c_1)}{u'(c_0)} R_j\right) = 1, \quad \forall j$$

where $R_j = \dfrac{x_j}{p_j}$. And we can also get that

$$\frac{\mathrm{E}(R) - R_f}{R_f} = -\mathrm{Cov}\left(\beta \frac{u'(c_1)}{u'(c_0)}, R\right)$$

Conclusion 1. *The marginal rate of substitution of* ***any unconstrained*** *investor is a valid SDF.*

With CRRA, we have:

$$\frac{\mathrm{E}(R) - R_f}{R_f} = -\mathrm{Cov}\left(\beta \left(\frac{c_1}{c_0}\right)^{-\gamma}, R\right)$$

B What is C?

Empirically, C is measured using some aggregated data, but aggregation is not an easy job. Suppose

$$C = \mathcal{C}\left[c\left(1\right), c\left(2\right), \cdots, c\left(m\right)\right].$$

The functional form of $\mathcal{C}$ is not observable. The FOC says

$$\mathrm{E}\left(\frac{\beta u'(\mathcal{C}\left[c_1(1), c_1(2), \cdots, c_1(m)\right])}{u'(C_0)} R_j\right) = 1, \quad \forall j,$$

but empirically, we measure $\mathcal{C}\left[c_1(1), c_1(2), \cdots, c_1(m)\right]$ by $C = \sum P_k c_k$.

Question: under what conditions, this measurement is valid? There are two related questions: how to measure economic growth? how to measure marginal utilities?

C Exchange rate

No arbitrage in home country (U.S.) implies that for all dollar denominated returns,

$$\mathrm{E}\left[mR\right] = 1. \tag{7}$$

No arbitrage in the foreign country (Europe) implies that for all Euro denominated returns,

$$\mathrm{E}\left[m^* R^*\right] = 1. \tag{8}$$

Here m and m^* are the SDF in US and Europe respectively.

Let x_0 denote the dollar per Euro exchange rate (that is, x_0 is the amount dollar can be exchanged per euro) at $t = 0$, and similarly, x_1 denotes the dollar per Euro exchange rate at $t = 1$. Note that equation (7) must hold for all dollar denominated returns. In particular, consider the following investment strategy: we start with \$1 on date 0. We convert it in to €$\frac{1}{x_0}$ and invest in a Euro-denominated return R^*. In period $t = 1$, we convert it back to dollar under exchange rate x_1, the return for this strategy is $\frac{1}{x_0} R^* x_1$. No arbitrage implies:

$$\begin{aligned} \mathrm{E}(m\frac{1}{x_0} R^* x_1) &= 1 \qquad (9) \\ \mathrm{E}(m^* R^*) &= 1 \qquad (10) \end{aligned}$$

Therefore, we must have for all Euro-denominated returns,

$$\mathrm{E}\left[m\frac{x_1}{x_0}R^*\right] = 1$$

Compare this with equation (8). If market is complete, we must have (prove this!):

$$m^* = m\frac{x_1}{x_0},$$

which means that

$$\frac{x_1(s)}{x_0} = \frac{m^*(s)}{m(s)}.$$

This is the basic equation that determines exchange rates in complete market models.

An intuitive way to think about the above equality is that $m^*(s)$ is the present value (measured in today's Euro) of one Euro paid in state s tomorrow. An alternative way to discount this payoff is to take that one Euro in state s, convert into US dollar to obtain $x_1(s)$ dollars, discount it using US SDF, and convert today's US dollar into $\frac{1}{x_0}$ Euro. The two ways to compute present value must be equivalent in a complete market, and hence $m^*(s) = m(s)\frac{x_1(s)}{x_0}$.

The Backus-Smith Puzzle

Assume CRRA utility with risk aversion γ

$$\frac{x_1(s)}{x_0} = \left(\frac{g^*}{g}\right)^{-\gamma}. \tag{11}$$

where g^* and g are the consumption growth rates in Europe and US respectively.

If Europe economy is a boom, then g^*/g will be high, hence $x_1(s)/x_0$ tends to be low, which means that Euro depreciates when Europe is in a boom and U.S. is in a recession. This is very much at odds with the data!

Brandt, Cochrane, Santa-Clara

The exchange rate equation implies:

$$\ln x_1(s) - \ln x_0(s) = \ln m^*(s) - \ln m(s) = \gamma(g - g^*) \tag{12}$$

Recall that that Hansen-Jagannathan Bound implies

$$\sigma(m) \geq \frac{1}{R_f}\frac{\mathrm{E}(R) - R_f}{\sigma(R)}$$

Assume that $\mathrm{E}(R) - R_f = 6\%$, and $\sigma(R) = 20\%$, then then HJ bound on the SDF is roughly

$$\frac{\sigma(m)}{\mathrm{E}(m)} \geq 30\%.$$

Using equation (12), we have:

$$\mathrm{Var}[\ln x_1 - \ln x_0] = \mathrm{Var}[\ln(m^*(s)] + \mathrm{Var}[\ln(m(s)] - 2\mathrm{Cov}(\ln m^*, \ln m).$$

In the data, $\left[\frac{1}{R_f}\frac{\mathrm{E}(R)-R_f}{\sigma(R)}\right]^2$ is much larger than $\mathrm{Var}[\ln x_1 - \ln x_0]$, which mean that the variance must be mostly cancelled by the covariance term. In other words, the correlation between $\ln m^*$ and $\ln m$ must be close to one. However, under standard CRRA,

$$\mathrm{Cov}(\ln m^*, \ln m) = \gamma^2 \mathrm{Cov}(\ln g, \ln g^*).$$

Consumption growth rates are not very correlated across countries.

Forward Premium Anomaly

Covered Interest Rate Parity

- Let $F_{0,T}$ be the \$/€ forward rate with maturity T. That is, consider a forward contract where you promise to pay Euro in one month in exchange for dollars. $F_{0,T}$ is the amount of dollar you can buy by paying Euro under such a contract. No arbitrage implies

$$\begin{aligned} F_{0,T} &= x_0 e^{(r-r^*)T} \\ \ln(F_{0,T}) &= \ln x_0 + rT - r^*T \end{aligned} \tag{13}$$

where x_0 is current exchange rate, r and r^* are log interest rates, and $F_{0,T}$ is forward exchange rate.

Conclusion 2. *CIP is a no-arbitrage relationship. It should hold well in the data for currencies that have liquid markets.*

Uncovered interest rate parity It refers to the expected version of (13):

$$\mathrm{E}_t(\ln x_{t+1}) = \ln x_t + r_t - r_t^*.$$

If UIP is violated, then carry trade can make a profit *on average*. Does it hold in the data?

To test whether uncovered interest rate parity holds, Fama runs the following regression,

$$\ln(x_{t+1}) - \ln(x_t) = \alpha + \beta(r_t - r_t^*) + \varepsilon$$

If the uncovered interest rate parity holds, then α should be close to 0, and β should be close to 1, but the empirical results show that β is close to 0, or $\beta < 0$.

Explaining the failure of UIP? The exchange rate equation implies:

$$\begin{aligned} \mathrm{E}_t\left[\ln(x_{t+1}) - \ln(x_t)\right] &= \mathrm{E}_t\left[\ln m_{t+1}^*\right] - \mathrm{E}_t\left[\ln_{m_{t+1}}\right] \\ &= \mathrm{E}_t\left[\ln m_{t+1}^*\right] - \ln \mathrm{E}_t\left[m_{t+1}^*\right] + \ln \mathrm{E}_t\left[m_{t+1}^*\right] \\ &\quad -\mathrm{E}_t\left[\ln_{m_{t+1}}\right] + \ln \mathrm{E}_t\left[m_{t+1}\right] - \ln \mathrm{E}_t\left[m_{t+1}\right] \end{aligned} \tag{14}$$

For any random variable x, we define $\mathcal{L}(x) = \ln \mathrm{E}[x] - \mathrm{E}[\ln x]$ to be entropy of x. Because ln is a concave function, we have

$$\mathcal{L}(x) \geq 0,$$

and "=" holds if and only if x is degenerate. Therefore, $\mathcal{L}(x)$ is a measure of how spread out x is. Kind of like variance.

With the definition of entropy, we can write (14) as

$$E[\Delta \ln x_{t+1}] = \mathcal{L}(m_{t+1}) - \mathcal{L}(m^*_{t+1}) + r_t - r^*_t, \tag{15}$$

where r_t denotes log (risk-free) interest rates.

What does this mean? Suppose for a minute that the entropy of the two SDF are equal $\mathcal{L}(m_{t+1}) - \mathcal{L}(m^*_{t+1}) = 0$ (for example, when there is no uncertainty). Equation (15) says that if US dollar interest is low, then US dollar must appreciate against Euro (on average): $E[\Delta \ln x_{t+1}] = r_t - r^*_t$.

In a world with uncertainty, the risk-free asset in the US is the claim that pays one unit of US consumption in all states of the world. The claim to US dollar is a claim that pays US consumption basket in all states of world. A low US interest rate means that the claim to one of unit of US consumption is expensive. That does not necessarily mean that the claim to one unit of US consumption is also expensive. As the consumption basket may be very risky. Equations (15) says even if $r_t < r^*_t$, $E[\Delta \ln x_{t+1}]$ may not be negative because we may have $\mathcal{L}(m_{t+1}) > \mathcal{L}(m^*_{t+1})$. That is, the US risk premium is high when r_t is low, making US consumption basket not a good asset to hold.

The above discussion implies that to account for the forward premium puzzle, we need a model of risk premium shocks.

Remarks on entropy: Using the no arbitrage equation,

$$1 = E[mR],$$

we have:

$$0 = \ln E[mR] \geq E[\ln m] + E[\ln R], \tag{16}$$

or

$$-E[\ln m] \geq E[\ln R].$$

Therefore,

$$\begin{aligned}\mathcal{L}(m) &= \ln E[m] - E[\ln m] \geq \ln E[m] + E[\ln R] \\ &= E[\ln R] - \ln R_f\end{aligned}$$

That is, the entropy of the SDF is bounded from below by

$$\mathcal{L}(m) \geq E[\ln R] - \ln R_f = EP,$$

where I simply defined

$$EP(R) = E[\ln R] - \ln R_f$$

to be the equity premium of R. This is just another way of provide a lower bound on the "variability" of the SDF.

The bound is achieved in (16). Assuming that this is the case, that is, we can find assets in home and foreign country such that the bound is achieved. Then:

$$E[\Delta \ln x_{t+1}] = EP\left(\hat{R}\right) - EP\left(\hat{R}^*\right) + r_t - r_t^*,$$

where $\hat{R}$ and $\hat{R}^*$ are the returns that achieves the bound. For a complete market model to replicate the forward premium puzzle, we basically need that whenever home country interest rate is high (relative to the foreign country), the home country risk premium must be low (relative to the foreign country). Therefore, interest rate variations does not affect exchange rate variation.

Remark 2. *(Entropy)*

Take any convex function f, we have

$$E[f(m)] - f(E[m]) \geq 0,$$

and "=" only if the distribution of m is degenerate.

We can think of $E[f(m)] - f(E[m])$ as a measure of how "dispersed" m is. Define $f(m) = -\ln m$, and

$$\mathcal{L}(m) = \ln E[m] - E[\ln m]$$

is the entropy of m. This is an important concept in information theory. The entropy bound is first derived by Bansal and Lehman.