Lecture 3: Consumption Based Asset Pricing

I Asset pricing with stand-in household

A The First Welfare Theorem

Consider an economy with two dates, 0 and 1. The commodity space of the economy is $\mathcal{C} = R \times X$, where X is the set of r.v.'s of the form: $x : (\Omega, \mathcal{F}, P) \to R$. There are K agents in the economy, each with endowment $e^k \in \mathcal{C}$. We also use e to denote the aggregate endowment. The preference of agent k is given by an utility function $V^k\left(c_0^k, c_1^k\right)$, where $c^k = \left(c_0^k, c_1^k\right) \in \mathcal{C}$.

An allocation $\left(c_0^k, c_1^k\right)_{k=1,\cdots K}$ is feasible if

$$\sum_{k=1}^{K} c^k \leq \sum_{k=1}^{K} e^k. \tag{1}$$

Suppose there are J securities, the payoff of the securities are $x_1, x_2, \cdots x_J$, with $x_j \in X$. Let $p = [p_1, p_2, \cdots p_J]^T$ be the vector of asset prices. Suppose also, there is no redundant asset, that is, the only vector $\alpha \in R^J$ that satisfies $\sum_{j=1}^{J} \alpha_j x_j = 0$ is $\alpha = 0$. A Walrasian equilibrium is a price vector p, an allocation $\hat{c} = \left(\hat{c}^k\right)_{k=1,\cdots K}$ a porfolio holding strategy for each agent, $\hat{h} = \left(\hat{h}^k\right)_{k=1,\cdots K}$ such that the following conditions are satisfied:

1. Utility maximization: $\forall k$,

$$\begin{aligned} \left(\hat{c}^k, \hat{h}^k\right) &\in \arg\max V^k\left(c_0, c_1\right) \\ s.t.\ c_0 + p^T h &\leq e_0^k \\ c_1(\omega) &\leq \sum_{j=}^{J} h_j x_j(\omega) + e_1^k(\omega)\text{, all } \omega \end{aligned}$$

2. Resource constraint: equation (1).

Theorem 1 *The First Welfare Theorem*

Suppose market is complete. Let $\left(\hat{c}^k\right)_{k=1,\cdots K}$ be a competitive equlibrium allocation with some p and $\hat{h}$. Suppose also, $\forall k$, $V^k(c_0, c_1)$ is strictly increasing in c_0. Then $\left(\hat{c}^k\right)_{k=1,\cdots K}$ is also Pareto Optimal.

For simplicity, we will assume Ω is finite dimensional; therefore $x \in X$ can be represented as a vector $x = [x_1, x_2, \cdots x_n]$. In this case, X is a $J \times n$ matrix of payoffs. Recall that the market is said to be complete if $\forall x \in X$, there exist an $h \in R^J$ such that $x = X^T h$. We first make the following observation:

Observation 1: Suppose market is complete, and there is no redundant asset, then there exists $q \in R^n$ such that $q >> 0$ and the budget constraint in the consumer's maximization problem is written as:

$$c_0 + \sum_{\omega=1}^{n} q(\omega) c_1(\omega) \leq e_0 + \sum_{\omega=1}^{n} q(\omega) e_1(\omega). \tag{2}$$

Exercise 1 *prove the above statement.*

B Implications of the First Welfare Theorem on Asset Pricing

Since we know CE must be PO, we solve asset pricing problems in economies that satisfies the assumption of first welfare theorems in two steps. First, we solve P.O. for the allocation; Second, we use the allocation to construct prices. We have the following convenient characterization of PO allocations.

Observation 2: PO allocation must solve the following optimization problem.

$$\begin{aligned} \left\{\hat{c}_0^k, \hat{c}_1^k\right\} &\in \arg\max \sum_{k=1}^{K} \lambda_k V\left(c_0^k, c_1^k\right) \\ \textit{subject to} &: \sum_{k=1}^{K} c^k \leq \sum_{k=1}^{K} e^k. \end{aligned}$$

The solution can be represented as $\hat{c}_0^k(e, \lambda), \hat{c}_1^k(e, \lambda)$. Once we solved for the allocations, we can use allocation to construct prices using the following observation:

Observation 3: The CE allocation must solve:

$$\begin{aligned} \left(\tilde{c}_0^k, \tilde{c}_1^k\right) &\in \arg\max V^k(c_0, c_1) \\ \textit{subject to} &: c_0 + \sum_{\omega=1}^{n} q(\omega) c_1(\omega) \leq e_0 + \sum_{\omega=1}^{n} q(\omega) e_1(\omega). \end{aligned}$$

This implies that the SDF has to satisfy:

$$m(\omega) = \frac{\frac{\partial}{\partial c_1(\omega)} V^k\left(\hat{c}_0^k, \hat{c}_1^k\right)}{\frac{\partial}{\partial c_0} V^k\left(\hat{c}_0^k, \hat{c}_1^k\right)}$$

for all k. That is, the marginal utility of any consumer can be used to construct the SDF.

Consider the following two special cases. First, if $V^k(c_0, c_1)$ is expected utility, that is,

$$V^k(c_0, c_1) = u(c_0) + \beta u(c_1).$$

In this case, Observation 3 implies

$$m(\omega) = \frac{\beta u'(c_1(\omega))}{u'(c_0)}. \tag{3}$$

This is the usual interpretation: prices are marginal utilities.

Next, we consider the special case of homothetic utility.

Observation 4: Suppose $V^k = V$ for all k and V is homothetic (definition), then

$$m(\omega) = \frac{\frac{\partial}{\partial c_1(\omega)} V^k\left(\hat{c}_0^k, \hat{c}_1^k\right)}{\frac{\partial}{\partial c_0} V^k\left(\hat{c}_0^k, \hat{c}_1^k\right)} = \frac{\frac{\partial}{\partial c_1(\omega)} V(e_0, e_1)}{\frac{\partial}{\partial c_0} V(e_0, e_1)}$$

does not depend on k.

CRRA is both expected utility and homothetic. In this case, stochastic discount factor is linked to consumption growth rates:

$$m(\omega) = \beta \left(\frac{c_1(\omega)}{c_0}\right)^{-\gamma}. \tag{4}$$

In general, (4) can be viewed as an approximation for (3). We leave this as a homework. Prove the following claim:

Observation 5: In the case of expected utility,

$$m(\omega) \approx \beta \left(\frac{c_1(\omega)}{c_0}\right)^{-\gamma},$$

where γ is the Arrow-Pratt measure of relative risk aversion.

II The Equity Premium Puzzle

Consider an economy where all agents have identical CRRA preferences: $u(c) = \frac{1}{1-\gamma}c^{1-\gamma}$. Let (c_0, c_1) denote the total endowment (which is also the total consumption) in the economy. Then by our previous result, the stochastic discount factor of the economy is:

$$m = \beta(\frac{c_1}{c_0})^{-\gamma} = \beta g^{-\gamma},$$

where $g = c_1/c_0$ is the consumption growth rate. The HJ bound put a lower bound on the volatility of any SDF given the set of asset prices we observe. In particular,

$$\sigma(m) \geq \frac{1}{\bar{r}}\frac{E[R_j] - \bar{r}}{\sigma(R_j)}$$

for all j. It turns out that in a large class of models, this also puts a lower bound on the risk aversion of the agent given data on the volatility of the consumption growth. It is important to note that HJ bound itself says nothing about risk aversion or volatility of consumption growth. It is the additional assumptions that we make in our models. However, it turns out that the additional assumptions is very general. That is, in a very general class of models, the risk aversion required for the HJ bound seems to be too high to be consistent with micro evidence. We turn to the additional assumptions now.

A Interpretation 1: Small Shocks in Consumption

Assume $g - E[g]$ is small, we can log-linearize around $E[g]$.

$$f(x) = f(\bar{x}) + f'(\bar{x})(x - \bar{x}) + o||x - \bar{x}||$$

Let $y = \ln x$, and consider a Taylor expansion around $\bar{y} = \ln \bar{x}$:

$$f(x) = f(e^y) = f(e^{\bar{y}}) + f'\left(e^{\bar{y}}\right)e^{\bar{y}}(y - \bar{y}) + o||y - \bar{y}|| = f(e^{\bar{y}})[1 + \frac{f'(x)\,x}{f(x)}(y - \bar{y}) + o||y - \bar{y}||$$

Note that

$$\frac{f'(x)\,x}{f(x)} = \frac{d\ln f(x)}{d\ln x}$$

is the elasticity of f with respect to x.

$$g^{-\gamma} \approx \bar{g}^{-\gamma}[1 - \gamma(ln(g) - ln(\bar{g})]$$

$$Var(g^{-\gamma}) \approx (\bar{g}^{-\gamma})^2\gamma^2 Var(ln(g))$$

$$\sigma(g^{-\gamma}) \approx (\bar{g}^{-\gamma})\gamma\sigma(ln(g))$$

Recall from the Mean Variance Frontier: $\sigma(m) \geq \frac{E[R_j - \bar{r}}{\sigma(R_j)}\frac{1}{\bar{r}}$
Therefore, using the values: $E[R_j] - \bar{r} = .07$, $\bar{r} = 1.02$, and $\sigma(R_j) = .02$, we find that $\sigma(m) \geq .35$. Since the data implies $\sigma(ln(g)) = .03$, we find that $\gamma \cdot .03 \geq .35$ or $\gamma > 12$. However, this level of risk aversion is far larger than is seen in the data.

B Interpretation 2: Log-Normality of Consumption Growth

We have not yet assumed a particular distribution of consumption growth; now we assume log normality.

$$ln(g) \sim N(\mu, \sigma^2)$$

$$\gamma ln(g) \sim N(\gamma\mu, \gamma^2\sigma^2)$$

$$Var(e^{-\gamma ln(g)}) = e^{-2\gamma ln\bar{g} + \gamma^2\sigma^2}(e^{\gamma^2\sigma^2} - 1)$$

$$\sigma\left(g^{-\gamma}\right) = e^{-\gamma ln(\bar{g}) + \frac{1}{2}\gamma^2\sigma^2}\sqrt{e^{\gamma^2\sigma^2} - 1} \approx \gamma\sigma$$

C The Habit model

There are mainly two kinds of habit model, internal habit (Constantinides (1990) and Sundaresan (1989)) and external habit (Campbell and Cochrane (1999)). For this section, we only talk about Campbell and Cochrane (1999) - the external habit formation model.

- The utility function of the representative agent takes the form

$$E\left[\sum_{t=0}^{\infty}\beta^t \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}\right]$$

where β is the discount rate, C_t is consumption, X_t is the external habit level and γ is the utility curvature parameter.

- Define the surplus consumption ratio

$$S_t \equiv \frac{C_t - X_t}{C_t}$$

where $0 < S_t < 1$.

- Relative risk aversion: if habit X_t is held fixed as fixed as consumption C_t varies, the coefficient of relative risk aversion is

$$\frac{-CU_{CC}}{U_C} = \frac{\gamma}{S_t},$$

which means the "real" risk aversion increases as surplus ratio S_t decreases, that is, as consumption approaches the habit level.

- The pricing kernel M_{t+1}:

$$M_{t+1} = \beta \frac{U_c\left(C_{t+1}, X_{t+1}\right)}{U_c(C_t, X_t)} = \beta \left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} \left(\frac{S_{t+1}}{S_t}\right)^{-\gamma}$$

- As you can see from pricing kernel and risk aversion, the process of S_t is crucial to asset pricing implications in the model. Campbell and Cochrane (1999) set the process of S_t so that the risk free rate is constant, which we call "reverse engineering".

D The rare disaster model

D.1 The Riez-Barro Model

Note that equity premium is puzzling under Normality assumptions, because SDF is a function of $\ln g$. If $\ln g$ is Normally distributed, then the moments of the SDF is completely determined by the first two moments of $\ln g$. So $Var\left(m\right)$ is directly tied to $Var\left(\ln g\right)$. This is not true under more general assumptions of the distribution of $\ln g$. Consider the following example.

Exercise: Suppose $ln(g) = ln(z) + N$ and $ln(z)$ is Normal distribution $N(ln(\bar{g}), \sigma^2)$. Then

$$N = \begin{cases} -D & with\, probability\ \frac{1}{2}\lambda \\ 0 & with\, probability\ 1 - \lambda \\ D & with\, probability\ \frac{1}{2}\lambda \end{cases}$$

Therefore$E[N] = 0$and $Var[N] = \frac{1}{2}\lambda D^2 + \frac{1}{2}\lambda D^2 = \lambda D^2$. Fix $\epsilon > 0$ and let $D = \sqrt{\frac{\epsilon}{\lambda}}$,

- Show that $Var(ln(g)) = \sigma^2 + \epsilon$.
- Show that $Var(m) \to \infty$as $\lambda \to 0$.

D.2 Fact-Tailed Distribution

Another way to break the link between $\sigma(\ln g)$ and $\sigma(m)$ is too assume fat-tailed distribution.

Exercise: (Weitzman) Suppose $ln(g)$ is a truncated t distribution with support $[-N, N]$. Fix $\sigma(\ln g)$, let $N \to \infty$, show that $\sigma(m) \to \infty$. That is, the equity premium can be as large as you want.

III Recursive utility and the LR model

A Recursive utility and preference for early resolution of uncertainty

A.1 The Kreps-Porteus example

Consider two economies with identical rep agents and identical consumption processes. The only difference is that in Economy L, the uncertainty of consumption in period 1 is resolved only in period 1, while the uncertainty in Economy E is resolved in period 0.

- Economy L: Late resolution of uncertainty

	Period 0		Period 1
π_H	C_0	Info	C_H
π_L		Arrive	C_L

(5)

- Economy E: Early resolution of uncertainty

		Period 0	Period 1
π_H	Info	C_0	C_H
π_L	Arrive	C_0	C_L

(6)

- In period -1, nature decides the consumption path, the probability is given by π_i, for $i = H, L$. In both economies, there is no uncertainty with respect to date 0 consumption. The only uncertainty is about date one consumption.

- In Economy L, the consumption path is not revealed to the agent until the end of peiod 1. In Economy E, the consumption path is revealed to the agent immediately. Krep and Porteus use a similar example to illustrate preference for early resolution.

- In the next sections, I briefly review the KP definition of early (late) resolution. I then derive the SDF of both economies. Finally, I prove that the in economy E, news announcement requires a positive premium if and only if agents prefers early resolution.

A.2 Preference for Early Resolution

We first set up some notation. A recursive preference can be described by a pair of aggregators, W and m. The function $m(\cdot)$ is called the certainty equivalence functional. In general, $m(\cdot)$ can be constructed using an strictly increasing and concave function h:

$$m(X) = h^{-1}\{E[h(X)]\}.$$

The KP preference features CRRA certainty equivalence functional, i.e. $h(X) = \frac{1}{1-\gamma}X^{1-\gamma}$. In this case,

$$m(X) = \left\{E\left[X^{1-\gamma}\right]\right\}^{\frac{1}{1-\gamma}}.$$

The function W is the intertemporal aggregator. KP uses the CES intertemporal aggregator:

$$W(c, m) = \left\{(1-\beta)\, c^{1-\frac{1}{\psi}} + \beta m^{1-\frac{1}{\psi}}\right\}^{\frac{1}{1-1/\psi}}.$$

Under the above notation, the date-0 utility of the rep agent of economy L can be calculated as:

$$U_L = W\left\{C_0, h^{-1}\left(E\left[h\left(C_1\right)\right]\right)\right\},$$

and the date-0 utility of the agent in economy E can be calculated as:

$$U_E = h^{-1}\{E[h \circ W\{C_0, C_1\}]\}.$$

Because h is strictly increasing, $U_L < U_E$ if and only if $h(U_L) < h(U_E)$, which is

$$h \circ W\left\{C_0, h^{-1}\left(E\left[h\left(C_1\right)\right]\right)\right\} < E\left[h \circ W\left\{C_0, C_1\right\}\right]. \tag{7}$$

Denote

$$x = h(C_1),$$

and

$$f(x) = h \circ W\left\{C_0, h^{-1}(x)\right\}. \tag{8}$$

Then the left-hand of (7) is

$$f(E[x]),$$

and the right-hand side of (7) is;

$$E[f(x)].$$

This implies that $U_L < U_E$ if and only if the f function defined in (8) is strictly convex. We make the following observation.

Claim 1 *(Preference for Early Resolution)*

The recursive preference (W, m) *exhibits preference for early (late) resolution of uncertainty if and only if the* f *function defined in (8) is strictly convex (concave).*

Under the KP preference, this is equivalent to $\gamma > \frac{1}{\psi}$ *(or* $\gamma < \frac{1}{\psi}$*).*

Note that under the KP preference,

$$f(x) = \frac{1}{1-\gamma}\left\{(1-\beta)\, C_0^{1-\frac{1}{\psi}} + \beta\left[(1-\gamma)\, x\right]^{\frac{1-1/\psi}{1-\gamma}}\right\}^{\frac{1-\gamma}{1-1/\psi}}.$$

One can show that

$$f''(x) > 0 \text{ if and only if } \gamma > \frac{1}{\psi}.$$

B The SDF for recursive utility

$$V(W_t) = \max_{C_t, \alpha_0}\left\{(1-\beta)\, C_t^{1-\frac{1}{\psi}} + \beta E\left[V\left(W_{t+1}\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}}\right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

$$\text{s.t.} \quad C_t + \alpha = e_t + W_t$$

$$W_{t+1} = \alpha R_{t+1} + e_{t+1}$$

where α is the share of the portfolio which agent buys at time t, R_{t+1} is the return for the portfolio at time $t+1$ and e_t, e_{t+1} is the endowment at time t and $t+1$ respectively.

Let us put the budget constraint into the utility function, we can get

$$V(W_t) = \left\{ (1-\beta)\left(e_t + W_t - \alpha\right)^{1-\frac{1}{\psi}} + \beta E\left[V\left(\alpha R_{t+1} + e_{t+1}\right)^{1-\gamma}\right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right\}^{\frac{1}{1-\frac{1}{\psi}}}$$

The first order condition of α :

$$(1-\beta)C_t^{-\frac{1}{\psi}} = \beta m^{-\left(\frac{1}{\psi}-\gamma\right)} E\left[V\left(W_{t+1}\right)^{-\gamma} V_W\left(W_{t+1}\right) R_{t+1}\right] \tag{9}$$

where $m = E\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$.

The envelop condition:

$$V_W = (1-\beta)\, V\left(W_t\right)^{\frac{1}{\psi}} C_t^{-\frac{1}{\psi}} \tag{10}$$

Combining equation (9) and equation (10), we can get

$$(1-\beta)C_t^{-\frac{1}{\psi}} = \beta m^{-\left(\frac{1}{\psi}-\gamma\right)} E\left[V\left(W_{t+1}\right)^{-\gamma}(1-\beta)\, V\left(W_{t+1}\right)^{\frac{1}{\psi}} C_{t+1}^{-\frac{1}{\psi}} R_{t+1}\right]$$
$$1 = E\left[\beta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{V\left(W_{t+1}\right)}{m}\right)^{\frac{1}{\psi}-\gamma} R_{t+1}\right]$$

Therefore, the pricing kernel $M_{t+1} = \beta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{1}{\psi}}\left(\frac{V_{t+1}}{E\left[V_{t+1}^{1-\gamma}\right]^{\frac{1}{1-\gamma}}}\right)^{\frac{1}{\psi}-\gamma}$

C The Mehra-Prescott model with recursive utility

See notes on the MP model with recursive utility