Lecture 6: Incomplete Markets

Part 1: Exogenously incomplete markets

I Early Attempts (Mankiw)

Consider an endowment economy with two dates, 0 and 1. Assume that there are a continuum of agents $k \in [0,1]$ and denote the aggregate endowment as (e_0, e_1). Assume each individual has utility: $u(c_0) + \beta E[u(c_1)]$. Denote agent k's endowment be $e^k = (e_0, e_1 + \epsilon_k)$ where ϵ_k is $i.i.d.$Also, ϵ_k is independent of e_1, and $E[\epsilon_k] = 0$. Compare two different market environments. In case A, Idiosyncratic Risk is uninsurable, while there are markets are complete in case B.

A The precautionary saving motive

Claim 1: Suppose $u'(\cdot)$is convex, then the interest rate is lower in the Economy with incomplete market. This is due to precautionary savings since $u'''(\cdot) > 0$.

We can prive the intuition of this result using a Taylor expansion argument.

$$\bar{r} = \frac{1}{E[m]} = \frac{1}{\beta \frac{E[u;(c_1)]}{u'(c_0)}}$$

Then

$$ln(\bar{r}) \approx -ln(\beta) + ln(u'(c_0)) - ln(Eu'(c_1))$$

We can then take a Taylor Series of $u(\cdot)$ around $c_1 = \bar{E}[c_1]$:

$$u'(c_1) = u'(\bar{c_1}) + u''(\bar{c_1})(c_1 - \bar{c_1}) + \frac{1}{2}u'''(\bar{c_1})(c_1 - \bar{c_1})^2 + O(c_1 - \bar{c_1})^3$$

$$E[u'(c_1)] = u'(\bar{c_1}) + \frac{1}{2}u'''(\bar{c_1})Var(c_1 - \bar{c_1}) = u'(\bar{c_1})[1 + \frac{1}{2}\frac{u'''(\bar{c_1})}{u'(\bar{c_1})}Var(c_1)]$$

Plugging this into the previous equation, we get:

$$ln(\bar{r}) = -ln(\beta) + ln(u'(c_0)) - [ln(u'(\bar{c_1})) + ln[1 + \frac{1}{2}\frac{u'''(\bar{c_1})}{u'(\bar{c_1})}Var(c_1)]]$$

$$ln(\bar{r}) \approx -ln(\beta) - [ln(u'(\bar{c_1})) - ln(u'(c_0)) - \frac{1}{2}\frac{u'''(\bar{c_1})}{u'(\bar{c_1})}Var(c_1)]$$

Assuming $u'''(\bar{c_1}) > 0$, we can see the agent has a higher level of precautionary savings.

I will use the notation $Du(e_1+\epsilon) = \beta\frac{u'(e_1+\epsilon)}{u'(e_0)}$ and $Du(e_1) = \beta\frac{u'(e_1)}{u'(e_0)}$. Additionally, I use $*$ to refer to variables in the complete market economy(case B). Recall that in a complete market (Market B) $q^* = E[m^*] = E[\beta\frac{u'(e_1)}{u'(e_0)}] = E[Du(e_1)]$. In the Incomplete Market (Market A), this is no long true. Instead we have:

$$q = E[m] = E[Du(e_1+\epsilon)] = E[E\{Du(e_1+\epsilon)\}|e_1] > E[Du(E[e_1+\epsilon|e_1]) = E[Du(e_1)] = E[m^*] = q^*$$

Therefore, $E[m] > E[m^*]$. The intuition behind this result is that due to the greater risk, agents are more willing to save, and therefore the risk-free rate must be lower in equilibrium.

Exercise 1 *Suppose the equity is the claim to a payoff $x\,(e_1)$, where x is an increasing function of e_1. Prove that the expected return on equity in economy A is also lower than that in economy B.*

B Equity premium

Consider an asset with payoff $x\,(e_1)$. Denote the price of the asset as p in the incomplete market economy and the price of the asset be p^* in the complete market economy. Also, as before denote q and q^* be the price of a one-period discount bond in the incomplete market and that in the complete market economy, respectively. Also denote $R_e = \frac{x(e_1)}{p} - \frac{1}{q}$ and $R_e^* = \frac{x(e_1)}{p^*} - \frac{1}{q^*}$ be the excess return in economy A and B.

Claim 3: Suppose that in the incomplete market economy, ε and x jointly satisfy $Var(\epsilon) = 0$ if $R_e > 0$ and $Var(\epsilon) > 0$ if $R_e \leq 0$. Also suppose $u'(c)$ is strictly convex. Then $\frac{E[x(e_1)/p]}{R_f} > \frac{E[x(e_1)/p^*]}{R_f^*}$. That is, the equity premium in the incomplete market economy is higher.

This is to say that to explain the equity premium puzzle in incomplete markets, we need to assume the variance of the idiosyncratic volatility ϵ is counter-cyclical!

Proof of Claim 3: We need to show $E[\frac{x(e_1)}{p}q] > E[\frac{x(e_1)}{p^*}q^*]$, which is equivalent to $\frac{q}{p} > \frac{q^*}{p^*}$. The intertemporal Euler equation implies:

$$E[Du(e_1+\epsilon)R_e] = 0 = E[Du(e_1)R_e^*]$$

Note that $qR_e = \frac{qx(e_1)}{p} - 1$. Therefore,

$$E[Du(e_1+\epsilon)(\frac{qx\,(e_1)}{p} - 1)] = 0 = E[Du(e_1)(\frac{q^*x\,(e_1)}{p^*} - 1)] \tag{1}$$

Note that $E[Du(e_1)(\frac{q^* x(e_1)}{p^*}-1)]$ implies $E[Du(e_1)x(e_1)]\frac{q^*}{p^*} = E[Du(e_1))]$. We now prove that $E[Du(e_1)x(e_1)]\frac{q}{p} > E[Du(e_1))]$ under our assumption of counter-cyclical volatility. Focusing on the incomplete market economy, the left half of equation (1) implies

$$\begin{aligned} 0 &= E[Du(e_1+\epsilon)(\frac{qx(e_1)}{p}-1)] = E[Du(e_1+\epsilon)qR_e] \\ &= Prob(R_e>0)E[Du(e_1+\epsilon)(\frac{qx(e_1)}{p}-1)|R_e>0] + Prob(R_e\le 0)E[Du(e_1+\epsilon)(\frac{qx(e_1)}{p}-1)|R_e\le 0] \\ &= Prob(R_e>0)E[Du(e_1)(\frac{qx(e_1)}{p}-1)|R_e>0] + Prob(R_e\le 0)E[E\{Du(e_1+\epsilon)|e_1\}(\frac{qx(e_1)}{p}-1)|R_e\le 0] \end{aligned}$$

By Jensen's inequalilty, the second term,

$$E[E\{Du(e_1+\epsilon)|e_1\}(\frac{qx(e_1)}{p}-1)|R_e\le 0] < Prob(R_e\le 0)E[Du(e_1)(\frac{qx}{p}-1)|R_e\le 0].$$

Therefore,

$$0 = E[Du(e_1+\epsilon)(\frac{qx(e_1)}{p}-1)] < E[Du(e_1)(\frac{qx}{p}-1)].$$

That is,

$$E[Du(e_1)x(e_1)]\frac{q}{p} > E[Du(e_1))],$$

as needed.

II Constantinides and Duffie

A Setup

Consider the following endowment economy.

- The growth rate of aggregate endowment Y is i.i.d. and has two possible realizations:

$$\frac{Y_{t+1}}{Y_t} = g, \quad where \quad g \in \{g_H, g_L\}. \tag{2}$$

- Individual i's endowment at time t is $y_{i,t} = Y_t \times s_{i,t}$ where

$$\ln s_{i,t} = \sum_{k=1}^{t} \varepsilon_{i,k}, \quad t = 0, 1, 2\cdots \tag{3}$$

  and $\varepsilon_{i,k}$ is i.i.d. over time with $E[e^{\varepsilon_{i,k}}] = 1$. That is, $y_{i,0} = Y_0$, $y_{i,t} = Y_1 \times e^{\varepsilon_{i,1}}$, $\cdots$.

Note that $E[y_{i,t}] = Y$. This is like the Constantinides-Duffie setup.

- If $g = g_H$, then the distribution of ε is degenerate. That is, $\varepsilon = 0$ with probability one.
- If $g = g_L$, then ε follows a negative exponential distribution, which captures the notion of "individual rare disasters".

Some facts about negative exponential distribution:Consider the following distribution:

$$f(x) = \begin{array}{cc} 0 & x > \bar{x} \\ \lambda e^{\lambda(x-\bar{x})} & x \leq \bar{x} \end{array}$$

Claim 1 *The integral of $f(x)$ is*

$$\int_{-\infty}^{y} f(x)\,dx = e^{\lambda(y-\bar{x})}.$$

In particular, $\int_{-\infty}^{\infty} f(x)\,dx = 1$ is a proper density. Also,

$$\int_{-\infty}^{y} e^{\theta x} f(x)\,dx = \frac{\lambda}{\lambda+\theta} e^{-\lambda\bar{x}+(\theta+\lambda)y} \quad for \quad \lambda+\theta > 0.$$

In particular,

$$\int_{-\infty}^{\infty} e^{\theta x} f(x)\,dx = \frac{\lambda}{\lambda+\theta} e^{\theta\bar{x}} \quad for \quad \lambda+\theta > 0.$$

With $\theta = 1$, the requirement $E[e^x] = 1$ requires that $\bar{x} = \ln\frac{1+\lambda}{\lambda}$. This is the parameter we assume.

B Stochastic Discount Factor

In order to derive the SDF, we first compute the utility of the agent. We guess $U_t = u(g_t)\,y_t = u(g_t)\,Y_t s_t$. Therefore, from $U = \left[(1-\beta)\,C^{1-\frac{1}{\psi}} + \beta\left(E\left[U'^{1-\gamma}\right]\right)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}$, we have:

$$u(g)\,Ys = \left[(1-\beta)\,(Ys)^{1-\frac{1}{\psi}} + \beta\left(E\left[\left(u\left(g'\right)Y's'\right)\right]\right)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}$$

Hence,

$$
\begin{aligned}
u_H &= \left[(1-\beta)+\beta\left\{\pi\left(u_H g_H\right)^{1-\gamma}+(1-\pi)\int\left(u_L g_L e^{\varepsilon}\right)^{1-\gamma} f(\varepsilon)\,d\varepsilon\right\}^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}, \\
u_L &= \left[(1-\beta)+\beta\left\{(1-\pi)\left(u_H g_H\right)^{1-\gamma}+\pi\int\left(u_L g_L e^{\varepsilon}\right)^{1-\gamma} f(\varepsilon)\,d\varepsilon\right\}^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}.
\end{aligned}
$$

We can also compute the certainty equivalent of the agent as

$$
\begin{aligned}
m_H &= \left\{\pi\left(u_H g_H\right)^{1-\gamma}+(1-\pi)\int\left(u_L g_L e^{\varepsilon}\right)^{1-\gamma} f(\varepsilon)\,d\varepsilon\right\}^{\frac{1}{1-\gamma}} \\
&= \left\{\pi\left[g_H u_H\right]^{1-\gamma}+(1-\pi)\frac{\lambda^{\gamma}(1+\lambda)^{1-\gamma}}{\lambda+1-\gamma}\left(g_L u_L\right)^{1-\gamma}\right\}^{\frac{1}{1-\gamma}},
\end{aligned}
$$

$$
m_L = \left\{(1-\pi)\left[g_H u_H\right]^{1-\gamma}+\pi\frac{\lambda^{\gamma}(1+\lambda)^{1-\gamma}}{\lambda+1-\gamma}\left(g_L u_L\right)^{1-\gamma}\right\}^{\frac{1}{1-\gamma}}.
$$

The marginal rate of substitution is:

$$
\begin{aligned}
\Lambda(g_H) &= \beta\left[\frac{s' g_H Y}{sY}\right]^{-\frac{1}{\psi}}\left[\frac{u_H s' g_H Y}{m_H Y}\right]^{\frac{1}{\psi}-\gamma} = \beta g_H^{-\frac{1}{\psi}}\left[\frac{u_H g_H}{m_H}\right]^{\frac{1}{\psi}-\gamma}, \\
\Lambda(g_L,\varepsilon) &= \beta\left[\frac{s' g_L Y}{sY}\right]^{-\frac{1}{\psi}}\left[\frac{u_L s' g_L Y}{m_L Y}\right]^{\frac{1}{\psi}-\gamma} = \beta\left[e^{\varepsilon} g_L\right]^{-\frac{1}{\psi}}\left[\frac{u_L e^{\varepsilon} g_L}{m_L}\right]^{\frac{1}{\psi}-\gamma} = \beta g_L^{-\frac{1}{\psi}}\left[\frac{u_L g_L}{m_L}\right]^{\frac{1}{\psi}-\gamma} e^{-\gamma\varepsilon}.
\end{aligned}
$$

Exercise 2 *Assume the market is incomplete and there are only two assets traded. The one-period risk-free bond, and the claim to aggregate consumption. In particular, assets with payoff contingent on the realizations of individual endowment are not traded.*

1. Show that a no-trading equililbrium exists. That is, in the no-trading competitive equilibrium, agents do not trade any asset and simply consume their endowment.

2. How is the risk-free interest rate in this economy compared with the one with complete market? Derive a pricing formula for aggregate state contingent claims.

The price of aggregate endowment can be calculated as follows. Suppose the price of aggregate wealth is of the form $p(g)Y$. Then the present value of the equity is calculated as:

$$p(g)\,Y = \pi(g, g_H)\,\beta g_H^{-\frac{1}{\psi}} \left[\frac{u_H g_H}{m_H}\right]^{\frac{1}{\psi}-\gamma} \left[p(g_H)\,Y' + Y'\right] + \pi(g, g_L)\,\beta g_L^{-\frac{1}{\psi}} \left[\frac{u_L g_L}{m_L}\right]^{\frac{1}{\psi}-\gamma} E\left[e^{-\gamma\varepsilon}\right]\left[p(g_L)\,Y' + Y'\right]$$

That is,

$$p(g) = \pi(g, g_H)\,\beta g_H^{-\frac{1}{\psi}} \left[\frac{u_H g_H}{m_H}\right]^{\frac{1}{\psi}-\gamma} \left[p(g_H) + 1\right] g_H + \pi(g, g_L)\,\beta g_L^{-\frac{1}{\psi}} \left[\frac{u_L g_L}{m_L}\right]^{\frac{1}{\psi}-\gamma} E\left[e^{-\gamma\varepsilon}\right]\left[p(g_L) + 1\right] g_L$$

Note that

$$E\left[e^{-\gamma\varepsilon}\right] = \frac{\lambda}{\lambda-\gamma}\left(\frac{\lambda}{1+\lambda}\right)^{\gamma} > 1.$$

Clearly, the presence of idiosyncratic risks raises the volatility of the SDF.

C The dual problem of utility maximization

Let $C(U, y)$ be the consumption policy for any agent with promised utility U and current level of endowment y. Consider the following profit maximization problem:

$$P(y, U, \phi) = \max_{C, \{U'(g',\varepsilon')\}_{g',\varepsilon'}} (y - C) + \sum_{g'=g_H, g_L} \pi(g')\, SDF\left(g' \middle| \phi\right) \sum_{\varepsilon'} \pi\left(\varepsilon' \middle| g'\right) P\left(y g' e^{\varepsilon'}, U(g', \varepsilon'), \phi'\right)$$

$$\textit{subject to}: \quad U = \left[(1-\beta)\, C^{1-\frac{1}{\psi}} + \beta \left(\sum_{g'=g_H, g_L} \pi(g') \sum_{\varepsilon'} \pi\left(\varepsilon' \middle| g'\right) U(g', \varepsilon')\right)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}$$

Note that homogeneity implies that $C(U, y) = c\left(\frac{U}{y}\right) y$ for any y for some function c. This motivates the definition of normalized utility $u = \frac{U}{y}$.

$$p(u, \phi) = \max_{c, \{u'(g',\varepsilon')\}_{(g',\varepsilon')}} (1 - c) + \sum_{g=g_H, g_L} \pi(g')\, \Lambda\left(g' \middle| \phi\right) \sum_{\varepsilon'} \pi\left(\varepsilon' \middle| g'\right) g' e^{\varepsilon'} p\left(u'(g', \varepsilon'), \phi'\right)$$

$$s.t: \quad u = \left[(1-\beta)\, c^{1-\frac{1}{\psi}} + \beta \left(\sum_{g} \pi(g) \sum_{\varepsilon'} \pi\left(\varepsilon' \middle| g'\right) \left[g' e^{\varepsilon'} u'(g', \varepsilon')\right]^{1-\gamma}\right)^{\frac{1-1/\psi}{1-\gamma}}\right]^{\frac{1}{1-1/\psi}}.$$

The optimality condition of the above problem are:

$$\frac{\partial}{\partial u}p\left(u|\,\phi\right) = -\lambda = -\frac{1}{\left(1-\beta\right)\left(\frac{c}{u}\right)^{-\frac{1}{\psi}}} \tag{4}$$

$$\frac{\beta}{1-\beta}c^{\frac{1}{\psi}}m\left(u\right)^{\gamma-\frac{1}{\psi}}\left(g'e^{\varepsilon'}u'\right)^{-\gamma} = -\Lambda\left(g'|\,\phi\right)\frac{\partial}{\partial u}p\left(u'\left(g',\varepsilon'\right)|\,\phi'\right), \tag{5}$$

where

$$m\left(u\right)=\left(\sum_{g'}\pi\left(g'\right)\sum_{\varepsilon'}\pi\left(\varepsilon'|\,g'\right)\left[g'e^{\varepsilon'}u'\left(g',\varepsilon'\right)\right]^{1-\gamma}\right)^{\frac{1}{1-\gamma}}$$

In the above economy, everybody's promised utility is $\bar{u}_gY$ and the cost of providing the utility is $\bar{p}_gY$, which is exactly the PV of the agent's endowment. The intermediary break even and maximizes utility. We can think of an agent whose endowment is e^{ε} fraction of the aggregate endownment, and the intermediary offered him a perfect risk sharing plan that delivers $ue^{\varepsilon}Y$ amount of utility. The cost of providing the utility is $\bar{p}_gY\times\frac{ue^{\varepsilon}Y}{\bar{u}_gY}=\frac{ue^{\varepsilon}}{\bar{u}_g}\bar{p}Y$, and the PV of the agent's endowment is $e^{\varepsilon}\bar{p}_gY$. Therefore, the profit for such a contract is:

$$e^{\varepsilon}\bar{p}_gY-\frac{ue^{\varepsilon}}{\bar{u}_g}\bar{p}_gY=e^{\varepsilon}\bar{p}_gY\left(1-\frac{u}{\bar{u}_g}\right).$$

Remember we use u to denote the promised utility to the agent normalized by his endowment, and $p_g\left(u\right)$ to denote the profit for the intermediary normalized by by his endowment. The above calculation implies that the normalized profit function (under perfect risk sharing) is

$$p_g^*\left(u\right)=\bar{p}_g\left(1-\frac{u}{\bar{u}_g}\right) \tag{6}$$

if we assume perfect risk sharing. Here we denote the profit function for perfect risk sharing as $p_g^*\left(u\right)$.

Under the perfect risk-sharing plan, for an agent whose promised utility is $\bar{u}\left(g\right)$, his future consumption will be $\{Y_{t=0,1,2,\cdots}\}$. Therefore, the current period consumption for any agenty with promised utility u is $\frac{u}{\bar{u}(g)}Y$. That is to say, $c_g^*\left(u\right)=\frac{\frac{u}{\bar{u}_g}Y}{Y}=\frac{u}{\bar{u}_g}$, for $g=g_H,g_L$.

Finally, note that from equation (6) that $\frac{\partial}{\partial}p_g^*\left(u\right)=-\frac{\bar{p}_g}{\bar{u}_g}$. Note that equation (4) implies that $\frac{\partial}{\partial}p_g^*\left(u\right)=-\frac{1}{(1-\beta)\left(\frac{c(g)}{u(g)}\right)^{-\frac{1}{\psi}}}$. Note that complete risk sharing implies $\frac{\bar{c}(g)}{\bar{u}(g)}=\frac{1}{\bar{u}(g)}$, where $\bar{c}\left(g\right)$ and $\bar{u}\left(g\right)$ are the normalized consumption and utility under first best. Therefore,

$\frac{\partial}{\partial}p_g^*(u) = -\frac{1}{(1-\beta)\bar{u}_g^{\frac{1}{\psi}}}$. It is not hard to prove that

$$\frac{\partial}{\partial}p_g^*(u) = \frac{\bar{p}_g}{\bar{u}_g} = -\frac{1}{(1-\beta)\,\bar{u}_g^{\frac{1}{\psi}}} \tag{7}$$

from the closed form solutions. Clearly, in the case of unit IES, we have

$$p_g^*(u) = \frac{1}{1-\beta}\left(1 - \frac{u}{\bar{u}_g}\right); \qquad \frac{\partial}{\partial}p_g^*(u) = -\frac{1}{1-\beta}\frac{1}{\bar{u}_g}, \quad for\ g = g_H, g_L$$