Lecture 6: Equilibrium Asset Pricing with Heterogenous Information

I Equilibrium Information Acquisition (Grossman and Stiglitz 1980)

Consider an economy with measure 1 of agents. All agents are identical and they are endowed with initial wealth w. All agents have the same CARA preference with constant Arrow-Pratt measure of absolute risk aversion parameter γ:

$$u(c) = -e^{-\gamma c}$$

There is a risky asset with uncertain payoff θ, where θ is of Normal distribution with mean $\bar{\theta}$ and variance ρ_θ^{-1}. There is a storage technology that allows the agents in the economy to transform one unit of wealth at date 0 to 1 unit of consumption goods at date 1 (That is, the risk-free return $\bar{r} = 1$). At date 0, an agent can choose to observe a common signal on θ at a cost κ:

$$\theta + \varepsilon,$$

where ε is a Normally distributed r.v. independent of θ with mean 0 and variance ρ_ε^{-1}. Therefore agents who choose to pay the cost and observe the signal faces the following optimization problem:

$$\max_x E\left[u\left\{(w - \kappa - px) + x\theta\right\} \mid \theta + \varepsilon\right],$$

where x denote the number of shares of the risky asset the agent choose to hold, and p is the price of the risky asset. Note that the expectations are taken conditioning on the common signal observed by the agents.

1. Suppose in equilibrium a fraction λ of the agents choose to observe the signal, prove that the total demand of all the informed agents (agents who choose to pay the cost and observe the signal), denote X_I is given by:

$$X_I = \frac{\lambda}{\gamma}\left[\rho_\theta \bar{\theta} + \rho_\varepsilon(\theta + \varepsilon) - (\rho_\theta + \rho_\varepsilon)p\right].$$

2. Suppose the equilibrium price also contains information about θ. Suppose observing

the equilibrium price is equivalent to observing the following signal on θ:

$$\theta + e,$$

where e is independent of θ, and $e \sim N\left(0, \rho_e^{-1}\right)$.* A trader who does not pay to observe the signal, $\theta + \varepsilon$ is called an uninformed trader, and a trader who choose to pay κ to observe the signal, $\theta + \varepsilon$ is called an informed trader. The mazximization problem of an uninformed trader is given by:

$$\max_x E\left[u\left\{(w - px) + x\theta\right\} \middle| \theta + e\right].$$

In this case, the total demand for the uninformed traders is:

$$X_U = \frac{1-\lambda}{\gamma}\left[\rho_\theta \bar{\theta} + \rho_e(\theta + e) - (\rho_\theta + \rho_e)p\right].$$

3. Suppose the total supply of the risky asset, denoted Q is a random variable:

$$Q = \bar{Q} + q,$$

where $q \sim N\left(0, \rho_q^{-1}\right)$, and $\bar{Q}$ is a constant. Assuming $\bar{Q} = 0$ for simplicity (that is, the total supply of the risky asset is zero on average). The market clearing condition is therefore:

$$X_I + X_U = \bar{Q} + q = q.$$

This gives:

$$\frac{\lambda}{\gamma}\left[\rho_\theta \bar{\theta} + \rho_\varepsilon(\theta + \varepsilon) - (\rho_\theta + \rho_\varepsilon)p\right] + \frac{1-\lambda}{\gamma}\left[\rho_\theta \bar{\theta} + \rho_e(\theta + e) - (\rho_\theta + \rho_e)p\right] = q,$$

which implies

$$\frac{1}{\gamma}\rho_\theta \bar{\theta} + \frac{\lambda}{\gamma}\rho_\varepsilon(\theta + \varepsilon) - q + \frac{1-\lambda}{\gamma}\rho_e(\theta + e) = \left[\frac{1}{\gamma}\rho_\theta + \frac{\lambda}{\gamma}\rho_\varepsilon + \frac{1-\lambda}{\gamma}\rho_e\right]p. \tag{1}$$

*This is a conjecture and will be verified later on.

Note that the term $\frac{\lambda}{\gamma}\rho_\varepsilon(\theta+\varepsilon)-q$ can be written as:

$$\begin{aligned} & \frac{\lambda}{\gamma}\rho_\varepsilon(\theta+\varepsilon)-q \\ = & \frac{\lambda}{\gamma}\rho_\varepsilon\left[\theta+\varepsilon-\frac{\gamma}{\lambda\rho_\varepsilon}q\right] \end{aligned}$$

Since both sides of (1) are $\theta+e$ measurable, $\theta+\varepsilon-\frac{\gamma}{\lambda\rho_\varepsilon}q$ must be $\theta+e$ measurable. The only possibility is

$$e=\varepsilon-\frac{\gamma}{\lambda\rho_\varepsilon}q. \tag{2}$$

With the above definition of e, the market clearing condition (1) is written as:

$$\frac{1}{\gamma}\rho_\theta\bar{\theta}+\frac{\lambda}{\gamma}\rho_\varepsilon[\theta+e]+\frac{1-\lambda}{\gamma}\rho_e(\theta+e)=\left[\frac{1}{\gamma}\rho_\theta+\frac{\lambda}{\gamma}\rho_\varepsilon+\frac{1-\lambda}{\gamma}\rho_e\right]p$$

Denote

$$\rho(\lambda)=\lambda\rho_\varepsilon+(1-\lambda)\rho_e, \tag{3}$$

we have:

$$\rho_\theta\bar{\theta}+\rho(\lambda)[\theta+e]=[\rho_\theta+\rho(\lambda)]p$$

Therefore the equilibrium price is given by:

$$p=\frac{\rho_\theta\bar{\theta}+\rho(\lambda)[\theta+e]}{\rho_\theta+\rho(\lambda)}=\xi\bar{\theta}+(1-\xi)[\theta+e], \tag{4}$$

where

$$\xi=\frac{\rho_\theta}{\rho_\theta+\rho(\lambda)}. \tag{5}$$

Equation 4 confirms our conjecture that observing the equilibrium price is equivalent to observing $\theta+e$, where e is given in 2. Therefore, equilibrium price also contains information about the unknown payoff θ. Solve for $E[\theta|p]$ and $Var[\theta|p]$. Is $Var[\theta|p]$ increasing or decreasing in the fraction of agents who choose to observe the common signal $\theta+\varepsilon$?

4. Suppose a rational expectations equilibrium exists with $\lambda\in(0,1)$. Given λ, there is a unique κ that makes the an investor indifferent between observing the signal at the cost κ and not observing the signal. Derive the expression for λ as a function of the parameters of the model.

5. (Bonus Question) Provide conditions on the cost κ such that an interior equilibrium with $\lambda \in (0,1)$ exists.

6. Suppose the condition in 5) holds. How does the informativeness of the price changes with ρ_q. Provide some intuition of your result.

Sketch of Solution:

We first recall the following useful results:

Lemma 1 *Suppose* $\theta \sim N\left(\bar{\theta}, \rho_\theta^{-1}\right)$, *and* $\varepsilon \perp \theta$ *and* $\varepsilon \sim N\left(0, \rho_\varepsilon\right)$, *then*

$$Var\left[\theta \middle| \theta + \varepsilon\right] = \left(\rho_\theta + \rho_\varepsilon\right)^{-1}$$

Lemma 2 *Suppose* $x \sim N\left(\mu, \Gamma\right)$, *then*

$$E\left[e^{-\frac{1}{2}x^T\Omega^{-1}x}\right] = \left(\frac{|\Omega|}{|\Gamma + \Omega|}\right)^{\frac{1}{2}} e^{-\frac{1}{2}\mu^T(\Gamma+\Omega)^{-1}\mu}$$

Consider the following general formulation of an investor's maximization problem:

$$\max_x E\left[-e^{-\gamma(w-\kappa+xp+x\theta)}\right]$$

Note

$$E\left[-e^{-\gamma(w-\kappa+xp+x\theta)}\right] = -e^{-\gamma[w-\kappa]-\gamma[E(\theta)-p]x+\frac{1}{2}\gamma^2 Var(\theta)x^2}$$

Maximizing over x, we have:

$$x = \frac{1}{\gamma}\frac{E\left(\theta\right) - p}{Var\left(\theta\right)}.$$

The value function is then written as:

$$V\left(w\right) = -e^{-\gamma(w-\kappa)-\frac{1}{2}\frac{[E(\theta)-p]^2}{Var(\theta)}},$$

where the expectation and variance are conditioning the information set of the investor.

We use I and U to denote the information set of the informed and uninformed investor. Therefore $U \subset I$. We want to find out the utility of being informed from the perspective of an uninformed investor. That is, we need to calculate

$$\begin{aligned} E\left[V_I(w)\middle| U\right] &= E\left[-e^{-\gamma(w-\kappa)-\frac{1}{2}\frac{[E(\theta|I)-p]^2}{Var(\theta|I)}}\middle| U\right] \\ &= -e^{-\gamma(w-\kappa)}E\left[e^{-\frac{1}{2}\frac{[E(\theta|I)-p]^2}{Var(\theta|I)}}\middle| U\right] \end{aligned}$$

Note conditioning on U, $E(\theta|I)$ is a Normal distribution

$$N\left(E\left[\theta|U\right], Var\left[\theta|U\right] - Var\left[\theta|I\right]\right).$$

Therefore, using the lemma, from the perspective of the uninformed investor,

$$E\left[e^{-\frac{1}{2}\frac{[E(\theta|I)-p]^2}{Var(\theta|I)}}\middle| U\right] = \frac{Var(\theta|I)}{Var(\theta|U)}e^{-\frac{1}{2}\frac{[E[\theta|U]-p]^2}{Var(\theta|U)}}$$

Therefore,

$$E\left[V(w)|z\right] = -e^{-\gamma(w-\kappa)} \times \frac{Var(\theta|I)}{Var(\theta|U)}e^{-\frac{1}{2}\frac{[E[\theta|U]-p]^2}{Var(\theta|U)}}$$

The expected utility of an uninformed investor is

$$V_U(w) = -e^{-\gamma w-\frac{1}{2}\frac{[E(\theta|U)-p]^2}{Var(\theta|U)}} = -e^{\gamma w} \times e^{-\frac{1}{2}\frac{[E[\theta|U]-p]^2}{Var(\theta|U)}}.$$

To the the uniformed investor prefer paying for being informed to staying uninformed, we must have:

$$\begin{aligned} e^{\gamma\kappa}\frac{Var(\theta|I)}{Var(\theta|U)} &\leq 1 \Longleftrightarrow \\ \kappa &\leq \frac{1}{\gamma}\ln\frac{Var(\theta|U)}{Var(\theta|I)}. \end{aligned}$$

Note that the information gain is a monotone function of $\frac{Var(\theta|U)}{Var(\theta|I)}$, where

$$Var(\theta|I) = (\rho_\theta + \rho_\varepsilon)^{-1},$$

and

$$Var(\theta|U) = (\rho_\theta + \rho_e)^{-1}, \tag{6}$$

where

$$\rho_e = \left(\rho_\varepsilon^{-1} + \frac{\gamma^2}{\lambda^2\rho_\varepsilon^2}\rho_q^{-1}\right)^{-1} = \rho_\varepsilon\left(1 - \frac{1}{1+\frac{\lambda^2}{\gamma^2}\rho_\varepsilon\rho_q}\right) \tag{7}$$

Hence

$$\frac{Var(\theta|U)}{Var(\theta|I)} = \frac{\rho_\theta+\rho_\varepsilon}{\rho_\theta+\rho_e} = \frac{\frac{\rho_\theta}{\rho_\varepsilon}+1}{\frac{\rho_\theta}{\rho_\varepsilon}+1-\frac{1}{1+\frac{\lambda^2}{\gamma^2}\rho_\varepsilon\rho_q}}$$

is a decreasing function of $\frac{\lambda^2}{\gamma^2}\rho_q$. The intuition is that the higher the fraction of informed investor, λ is, the more informative the price will be (as

$$Var[e] = Var\left[\varepsilon - \frac{\gamma}{\lambda\rho_\varepsilon}q\right] = \rho_\varepsilon^{-1} + \frac{\gamma^2}{\lambda^2\rho_\varepsilon^2}\rho_q^{-1} \tag{8}$$

is smaller). Therefore the gain from observing the true signal will be smaller.

To answer question 3), note $Var[\theta|p] = Var[\theta|U]$ and is given by (6) and (7). Also,

$$\begin{aligned} E[\theta|p] &= \frac{\rho_\theta}{\rho_\theta+\rho_e}\bar{\theta} + \frac{\rho_e}{\rho_\theta+\rho_e}(\theta+e) \\ &= \frac{\rho_\theta}{\rho_\theta+\rho_e}\bar{\theta} + \frac{\rho_e}{\rho_\theta+\rho_e}\left[\frac{p-\xi\bar{\theta}}{1-\xi}\right], \end{aligned}$$

where ξ is given by (3) and (5).

For question 4), for a given λ, the indifference condition implies

$$\kappa = \kappa^*(\lambda), \quad where \quad \kappa^*(\lambda) = \frac{1}{\gamma}\ln\frac{\frac{\rho_\theta}{\rho_\varepsilon}+1}{\frac{\rho_\theta}{\rho_\varepsilon}+1-\frac{1}{1+\rho_\varepsilon\rho_q\lambda^2/\gamma^2}} \tag{9}$$

is a strictly decreasing function, and $\kappa^*(0) = \frac{1}{\gamma}\ln\frac{\rho_\theta+\rho_\varepsilon}{\rho_\theta}$, and $\kappa^*(1) = \frac{1}{\gamma}\ln\frac{\frac{\rho_\theta}{\rho_\varepsilon}+1}{\frac{\rho_\theta}{\rho_\varepsilon}+1-\frac{1}{1+(\rho_\varepsilon\rho_q)/\gamma^2}}$. Consequently, if $\kappa \in (\kappa^*(1), \kappa^*(0))$, then interior equilibrium exists.

Clearly, equation (9) also defines λ as a function of κ. For a given κ, in equilibrium, optimal information acquisition implies

$$\frac{Var(\theta|I)}{Var(\theta|U)} = e^{-\gamma\kappa}.$$

Therefore, the informativeness of the price, measured by $Var(\theta|U)$ is

$$Var(\theta|U) = e^{\gamma\kappa}Var(\theta|I) = \frac{e^{\gamma\kappa}}{(\rho_\theta+\rho_\varepsilon)} = \frac{\rho_\theta e^{\gamma\kappa}}{(1+\rho_\varepsilon/\rho_\theta)}.$$

From equation (8), the informativeness of the price is increasing in ρ_q. This is intuitive. The smaller the noise contained in the supply is, the more informative the equilibrium price will be.

To summarize:

1. The higher the fraction of the informed agents, the more informative is the price system.

2. The more individuals who choose to be informed (hence the more informative the price will be), the smaller is the gain from being informed.

3. A decrease in risk aversion leads informed traders to take larger positions, and and this increases the informativeness of the price system (equation (8)).

4. The equilibrium informativeness of the price system depends only on γ, κ, and $\rho_\varepsilon/\rho_\theta$. Changes in other parameters are irrelevant. For example, a decrease in ρ_q increases the noise in agg supply. For a fixed λ, this makes the equilibrium price less informative. However, this also increases the information gain for being informed. Consequently, more agents will choose to be information, and the two effects offsets each exactly.

5. The higher the cost of being informed, the less amount of people will choose to be informed (equation (9)).

6. As the quality of the informed trader's information improves (ρ_ε increases), the price will be more informative (equation (7)). However, $\lambda^*(\kappa)$ may or may not increase, because even though the value of being informed increases with ρ_ε, the value of being uninformed also increases as the price system becomes more informative (equation (9)).

7. In the limit, if there is no noise ($\rho_q \to \infty$), price conveys all information ($U = I$, equation (8)), and there is no incentive to purchase any information (no info is the only possible eq). But if every is uninformed, then it clearly pays some to get informed (no info cannot be eq either). Therefore, there can be no equilibrium.

8. What is the difference between the demand of the informed trader and that of an average trader?

$$
\begin{aligned}
X_I - q &= \frac{1}{\gamma}\frac{E(\theta) - p}{Var(\theta)} - q \\
&= \frac{1-\lambda}{\gamma}\left[\rho_\varepsilon(\theta+\varepsilon) - \rho_e(\theta+e) - (\rho_\varepsilon - \rho_e)p\right] \to 0
\end{aligned}
$$

as $\lambda \to 0$ or 1. Intuition: little of close to perfect information means everyone's belief are similar, therefore there is little trade. This happens when κ is close to $\kappa^*(1)$ or $\kappa^*(0)$.

II Noisy Rational Expectation Equilibrium with Correlated Errors

A Preferences and Production Technology

Here, we consider an economy populated by measure 1 of agents, and we index them as $i \in [0,1]$. All agents are identical and they live for two periods, 0 and 1. All agents have identical intertemporal preferences represented by:$\ln c_0 + \beta E^i [\ln c_1]$ .In period 0, each agent is endowed with identical initial amount of capital, K_0.

Capital is the only factor of input, and consumption goods are produced from capital via an AK technology:

$$Y_t = A_t K_t, \quad t = 0, 1.$$

There is no uncertainty at date 0, and we denote $A_1 = e^a$, where a is Normally distributed: $N\left(0, \rho_a^{-1}\right)$.

B Notion of Equilibrium

There are two types of assets traded, capital, and a risk-free bond. Denote the risk-free interest rate by e^r. The maximization problem of an individual consumer is written as:

$$\begin{aligned} & \max \ln c_0^i + \beta E^i \left[\ln c_1^i\right] \\ c_0^i + B^i + qK^i &= p_0 K_0 \\ c_1^i &= B^i e^r + e^a K^i \end{aligned}$$

Here the interpretation is that each agent owns K_0 on date 0 and p_0 is the cum-dividend price of capital. We assume that the price of date-1 capital is q, and use K^i denote the amount of date-1 capital held by consumer i. The rest of the budget constraint can be easily interpreted.

Firm's problem at date 0:

$$\max \{D_0 + qK_1 - p_0 K_0\}$$
$$\begin{aligned} D_0 &= A_0 K_0 - H(I, K_0) \\ K_1 &= (1-\delta) K_0 + I \end{aligned}$$

The function $H(I, K)$ is typically called the adjustment cost function. We assume that H is homogenous of degree 1 so that it can be represented as $H(I, K) = h\left(\frac{I}{K}\right) K$. Note that the optimization problem can be written as

$$\max \{A_0 K_0 - H(I, K_0) + q[(1-\delta) K_0 + I] - p_0 K_0\}$$

First order condition with repect to K_0 gives

$$p_0 = A_0 - H_K(I, K) + q(1-\delta).$$

First order condition with respect to I gives

$$q = H_I(I, K_0).$$

The resource constraint/market clearing conditions are given by:

$$\begin{aligned} \int B^i di &= 0 \\ \int K^i di &= K_1 \\ \int c_0^i di &= A_0 K_0 - H(I, K_0) \\ \int c_1^i di &= e^a K_1 \end{aligned}$$

If agents all have identical belief systems, then it is easy to see that the above economy boil down to the standard neoclassical rep agent model. We first consider the solution to the neoclassical model.

C The Neoclassical Model

In this case, we first solve the social planner's problem and use the PO allocation to construct prices.

$$
\begin{aligned}
& \max \ln c_0 + \beta E\left[\ln c_1\right] \\
C_0 + H\left(I, K_0\right) &= A_0 K_0 \\
C_1 &= e^a\left[\left(1-\delta\right) K_0 + I\right]
\end{aligned}
$$

Exercise: Show that the investment decision does not depend on the distribution of a. In particular, optimal investment is give by:

$$I = iK,$$

where i is the solution to:

$$\frac{h'\left(i\right)\left[\left(1-\delta\right)+i\right]}{A_0 - h\left(i\right)} = \frac{\beta}{1-\beta}$$

The interpretation is that the consumption-to-wealth ratio is constant. In what follows, we assume asymmetric information and correlated errors. We also assume no adjustment cost for simplicity.

D Information and Belief Systems

We assume each agent observe a private signal of the true productivity shock. In addition, we assume the equilibrium price is NOT fully revealing, and observing the equilibrium price is equivalent to observing a noisy signal of the true productivity (to be verified). In particular, for every agent i, we assume the information of a comes from two sources:

- **Private Information**:
  Each agent has a private information on the future realization of a, in the form of $s_i = a + \epsilon_i$, and
  $$s_i \sim N(a, \rho_s^{-1}).$$

- **The Market Price (Interest Rate)**:
  We also assume, and later on verify that observing the equilibrium market price is equivalent to observing a signal z, where
  $$z \sim N(a, \rho_z^{-1}).$$

We now describe how agent update their beliefs about a.

First, consider the standard Bayesian updating. The prior of the distribution of a is

$$a \sim N(0, \rho_a^{-1}),$$

then the agent will update his belief about a, to the distribution of

$$a|s^i, z \sim N((\rho_a + \rho_z + \rho_s)^{-1}[\rho_a \cdot 0 + \rho_z \cdot z + \rho_s \cdot s^i], (\rho_a + \rho_z + \rho_s)^{-1}),$$

write $\rho = \rho_a + \rho_z + \rho_s$ for short,

$$a|s^i, z \sim N(\rho^{-1}[\rho_a \cdot 0 + \rho_z \cdot z + \rho_s \cdot s^i], \rho^{-1}).$$

Second, we assume agents make small correlated errors. In particular, we assume that we assume the posterior mean of a is $\hat{a}^i = E[a|s^i, z] + \tilde{\varepsilon}$ where $\tilde{\varepsilon} \sim N(0, \rho_{\varepsilon}^{-1})$ and $\tilde{\varepsilon}$ is independent of any other random varibles. It is important to note here that agents are not fully rational and are making the same mistakes.[†]

$$\begin{aligned}
a|s^i, z &\sim N(\hat{a}^i, \rho^{-1}) && (10)\\
&\sim N(E[a|s^i, z] + \tilde{\varepsilon}, \rho^{-1}) && (11)\\
&\sim N(\rho^{-1}[\rho_a \cdot 0 + \rho_z \cdot z + \rho_s \cdot s^i] + \tilde{\varepsilon}, \rho^{-1}). && (12)
\end{aligned}$$

Remark 1 *Let us make two remarks here.*

- *Why we add this $\tilde{\varepsilon}$-mistake? Mechanically, we need something to prevent equilibrium price to be fully revealing. When the derivation goes further, you will see that if we do not add this $\tilde{\varepsilon}$-mistake, then the market price r would be so informative that agents can back up the true a in the equilibrium, and thus wash out the uncertainty part of the model.*

- *Does the $\tilde{\varepsilon}$-mistake make economic sense? The interpretation is that this is the COMMON MISTAKE made by all agents in the economy. They have to be COMMON, otherwise, their effect will wash out in the aggregate. They have to be mistakes, otherwise, the agent should use the Bayes rule, and the problem disappears. It is up to you to determine whether this is something that makes sense.*

[†]The agents understand that they are making mistakes. However, they do not know the exact amount of mistake, namely, the value of $\tilde{\varepsilon}$.

Up to now, I finished the description of the economy and the equilibrium concept. We however, need to construct the equilibrium, to verify that there is one, and to study it properties.

E Construction of Equilibrium

E.1 Individual's Optimization Problem

We first solve the optimal portfolio choice of an agent for a given belief system, $N\left(\mu, \rho^{-1}\right)$. For each agent $i \in [0,1]$, the individual's optimization problem can be summarized into the following:

$$\max_{C_0^i, I^i, B^i} \{(1-\beta)\ln C_0^i + \beta \hat{E}^i[\ln C_i^i]\} \tag{13}$$

$$s.t. \qquad C_0^i + I^i + B^i = K_0$$

$$C_1^i = e^a\ [\lambda K_0 + I^i] + RB^i$$

Here, we write $\hat{E}^i[\ \cdot\ ]$ as a shorthand for the expectation with respect to the distribution in (10).

Exercise: Prove that the optimal solution to the agent's problem can be completely characterized by the following conditions:

$$C_0^i = (1-\beta)(A_0 + 1 - \delta)K_0 \tag{14}$$

$$B^i = (1-\omega^i)\beta(A_0 + 1 - \delta)K_0 \tag{15}$$

$$I^i + \lambda K_0 = \omega^i \beta(A_0 + 1 - \delta)K_0 \tag{16}$$

$$1 = \hat{E}^i[\frac{1}{(e^{a-r}-1)\omega^i + 1}]^{\ddagger} \tag{17}$$

There is a very simple interpretation of the above conditions.

Note once we use condition (17) to solve for the ω^i for each i, the solution to household's optimization problem is completely characterized. Note for each ω, (17) is the expectation of a nonlinear function of $a - r$. Because Normal distribution depends only on it first two moments, (17) implicitly defines ω^i as a function of the first two moment of $a - r$. We denote $\omega^i = \omega(\hat{a}^i - r)$. Here we suppress the second moment, just because it is the same of

‡In case you forgot, we assumed before that R is the gross risk-free rate which can also be defined as $r = \ln R$.

all agents.

Exercise: Compute numerically $\omega(\hat{a}^i - r)$ for

$$\hat{a}^i - r \in [-10\%, \quad 10\%],$$

assuming $\rho = 300$.

Now we are ready to close the model by imposing the market clearing condition.

F The Market Clearing Conditions

In order to solve for a well-defined equilibrium, we need any equilibrium satisfy the following market clearing conditions:

$$\begin{aligned}
\text{Period-0 Commodity Market} &: \int C_0^i + I^i \; di = K_0 \\
\text{Bond Market} &: \int B^i \; di = 0 \\
\text{Period-1 Commodity Market} &: \int C_1^i \; di = e^a \; [\lambda K_0 + \int I^i \; di]
\end{aligned}$$

Exercise: Prove the above market clearing conditions are equivalent to the following

$$\int \omega(\hat{a}^i - r) \; di = 1. \tag{18}$$

Note the market clearing condition states that ω^i has to sum (integrates) to 1. Note that ω^i is a function of $\hat{a}^i - r$, and

$$\begin{aligned}
\int \hat{a}^i \; di &= \int \{E[a|s^i, z] + \tilde{\varepsilon}\} \; di \\
&= \int \{\rho^{-1}[\rho_a \cdot 0 + \rho_z \cdot z + \rho_s \cdot s^i] + \tilde{\varepsilon}\} \; di \\
&= \frac{\rho_z}{\rho} z + \frac{\rho_s}{\rho} \int s^i \; di + \tilde{\varepsilon} \qquad (\int s^i \; di = \int (a + \epsilon_i) \; di = a + \int \epsilon_i \; di = a) \qquad (19) \\
&= \frac{\rho_z}{\rho} z + \frac{\rho_s}{\rho} a + \tilde{\varepsilon} \\
&= \frac{\rho_z}{\rho} z + \frac{\rho_s}{\rho} (a + \frac{\rho}{\rho_s} \tilde{\varepsilon})
\end{aligned}$$

Then,

$$\begin{aligned}\hat{a}^i - \int \hat{a}^i \ di =& \frac{\rho_s}{\rho}(s^i - a) \\ \hat{a}^i \sim& N(\int \hat{a}^i \ di, (\frac{\rho_s}{\rho})^2 \times \rho_s^{-1}) \\ \sim& N(\int \hat{a}^i \ di, \frac{\rho_s}{\rho^2})\end{aligned} \tag{20}$$

This naturally leads to

$$\hat{a}^i - r \sim N(\int \hat{a}^i \ di - r, \frac{\rho_s}{\rho^2}). \tag{21}$$

Because the total measure of agents is 1. By law of large numbers, the integral in (18) can be interpreted as expectation. By equation (21), $\hat{a}^i - r$ is a Normal distribution. This implies that for any r, the LHS of equation (18), $E\left[\omega(\hat{a}^i - r)\ \right]$ is a function of $\left(\int \hat{a}^i \ di - r, \frac{\rho_s}{\rho^2}\right)$. Because $\frac{\rho_s}{\rho^2}$ does not depend on the realization of a and $\tilde{\varepsilon}$, neither will $\int \hat{a}^i \ di - r$. So our first observation is that $\int \hat{a}^i \ di - r$ has to be a constant. That is,

$$\frac{\rho_z}{\rho} z + \frac{\rho_s}{\rho}\left(a + \frac{\rho}{\rho_s}\tilde{\varepsilon}\right) - r$$

must be a constant. There maybe multiple construction of the equilibrium, but here is a straightforward one:

$$z = a + \frac{\rho}{\rho_s}\tilde{\varepsilon}; \quad r = \frac{\rho_z + \rho_s}{\rho} z - cons, \tag{22}$$

where $cons$ is some constant.

Under the above construction, $\int \hat{a}^i \ di - r = cons$, and (18) can be written as;

$$E\left[\omega\left(x\right)\right] = 1, \tag{23}$$

where x is Nomal distribution with $N\left(cons, \frac{\rho_s}{\rho^2}\right)$.

Please assume $\rho_a = \rho_s = \rho_z = 100$. (What does this mean for the distribution of $\tilde{\varepsilon}$?)

Exercise: Plot interest rate $r\left(z\right)$ as a function of z.

Hint: Use equation (23) to calculate $cons$, and use equation (22) to get

$$r\left(z\right) = \frac{\rho_z + \rho_s}{\rho} z - cons = \frac{2}{3} z - cons.$$

Exercise: what is the risk premium in this economy? How does risk premium depend on z?

Hint: Risk premium is

$$E[a - r(z)|z] = z - r(z) = cons + \frac{\rho_a}{\rho} z$$

and is increasing in z. The intuition is that when z goes up, the agent's belief about a goes up by less than 100% (Bayes rule, you do not put all weight on new information). Therefore interest rate goes up only partially. Consequently, $z - r(z)$ is increasing in z—because $\frac{\partial r(z)}{\partial z} < 1$. This is just the consequence of Bayes rule.

Exercise: The **Aggregate Investment** of this economy is

$$\int I^i \, di.$$

Is the aggregate investment increasing or decreasing in z?

G A Few Words on Numerical Solution:

To evaluate $E[\omega(x)]$ in equation (23), we need to evaluate the expectation of a nonlinear function of a Normally distributed random variable with mean *cons*, and variance $\frac{\rho_s}{\rho^2} = \frac{1}{900}$. Pick a vector of points $[x_1, x_2, \cdots x_n]$ with $x_{j+1} - x_j = h$. We can compute the integral approximately by

$$E[\omega(x)] = \sum_{j=1}^{n} \int_{x_j - h}^{x_j + h} \phi\left(t | \, cons, \quad \frac{1}{900}\right) \omega(x_j) \, dt,$$

where $\phi\left(\cdot | \, cons, \quad \frac{1}{900}\right)$ is the density of the standard Normal distribution with mean *cons* and variance $\frac{1}{900}$. Therefore, once we have $\omega(x)$ at the n points, $x_1, x_2, \cdots x_n$, we can compute the integral for every *cons* and search for the one that solves the market clearing condition. Once we solve for the *cons* we need to make sure that the n points covers at least $\pm std$ from *cons* to make sure that the approximation is accurate.

Now the problem becomes how to solve the $\omega(x)$ on a set of grid points. We need to solve

$$1 = E[\frac{1}{(e^X - 1)\omega + 1}], \tag{24}$$

where X is Normal distribution with $N\left(x, \frac{1}{300}\right)$.