Lecture 1: Misallocation Basics: The Static Setup

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Summary: This lecture discussion modeling and measuring misallocation in static setups. We focus on three issues: measurement, aggregation, and information.

I Theory ahead of measurement

The monopolistic competition setup Hsieh and Klenow (2009) develops a framework for modelling and measuring misallocation. This framework uses the monopolistic competition setup of Melitz(2003). The final good is produced from a continuum of intermediate input using the DS production technology:

$$Y = \left[\int_{[0,1]} y_j^{\frac{\eta-1}{\eta}} dj\right]^{\frac{\eta}{\eta-1}}, \tag{1}$$

where η is the elasticity of substitution across varieties. The final good producer is perfectly competitve. Therefore, given price p_j, the demand function of the final goods producer is $\frac{y_j}{Y} = p_j^{-\eta}$. Clearly, η is the elasticity of substitution across varieties: for 1% increase in prices, demand reduce by η%.

The intermediate goods producers are monopolists. They fully take into account the impact of their production decison on prices and maximize profit subject to prices distortions:

$$\begin{aligned} \pi(j) &= \max_{k_j,l_j} \{(1-\tau_j) p_j y_j - MPK_j k_j - MPL \times l_j\} \\ \textit{subject to} &: \quad p_j = \left[\frac{y_j}{Y}\right]^{-\frac{1}{\eta}} \\ y_j &= A_j k_j^{\alpha} l_j^{1-\alpha}. \end{aligned}$$

Here MPK_j is the firm-specific rental rate of capital, which Hsieh and Klenow denote as $R(1+\tau_{K,j})$, and $(1-\tau_j)$ is the output distortion. The demand funtion $p_j = \left[\frac{y_j}{Y}\right]^{-\frac{1}{\eta}}$ requires some comment. p_j is the price of variety j measured in composite commodity units. That is, we normalize the price so that $\int p_j y_j = Y$.

We denote

$$TR_j = (1-\tau_j) p_j y_j$$

to be the total revenue of firm j. We first establish the following lemma.

Lemma 1 *(Aggregate TFP with distortions) In the above setup,*

1. *The capital share, labor share and profit share are constant across all firms:*

$$\frac{MPK_j k_j}{(1-\tau_j) p_j y_j} = \alpha\left(1-\frac{1}{\eta}\right), \quad \frac{MPL \times l_j}{(1-\tau_j) p_j y_j} = (1-\alpha)\left(1-\frac{1}{\eta}\right), \quad \frac{\pi_j}{(1-\tau_j) p_j y_j} = \frac{1}{\eta}. \tag{2}$$

2. The aggregate production function can be written as $Y = TFP \times K^{\alpha} L^{1-\alpha}$, where

$$TFP = \frac{\left\{\int \left[\frac{(1-\tau_j)A_j}{(MPK_j/R)^{a}}\right]^{\eta-1} dj\right\}^{\frac{\eta}{\eta-1}}}{\left[\int \frac{(1-\tau_j)^{\eta} A_j^{\eta-1}}{(MPK_j/R)^{1+\alpha(\eta-1)}} dj\right]^{\alpha} \left[\int \frac{(1-\tau_j)^{\eta} A_j^{\eta-1}}{(MPK_j/R)^{\alpha(\eta-1)}} dj\right]^{1-\alpha}} \tag{3}$$

Proof. *Note that firm j's profit optimization problem can be written as:*

$$\max_{k_j, l_j} \left\{ (1-\tau_j) \left[A_j k_j^{a} l_j^{1-\alpha}\right]^{1-\frac{1}{\eta}} Y^{\frac{1}{\eta}} - MPK_j k_j - MPL \times l_j \right\}.$$

The first order conditions for k_j implies:

$$(1-\tau_j)\,\alpha \left(1 - \frac{1}{\eta}\right) y_j^{-\frac{1}{\eta}} Y^{\frac{1}{\eta}} A_j k_j^{a-1} l_j^{1-\alpha} = MPK_j. \tag{4}$$

Note that $y_i^{-\frac{1}{\eta}} Y^{\frac{1}{\eta}} A_j k_j^{a} l_j^{1-\alpha} = p_j y_j$, therefore the above can be written as:

$$(1-\tau_j)\,\alpha \left(1 - \frac{1}{\eta}\right) p_j y_j = MPK_j k_j. \tag{5}$$

Similarly, the first order condition on l_j can be used to show

$$(1-\tau_j)(1-\alpha)\left(1 - \frac{1}{\eta}\right) p_j y_j = MPL l_j. \tag{6}$$

This proves (2).

Note that (5) and (6) implies that capital-to-labor ratio can be expressed as a function of wedges:

$$\frac{k_j}{l_j} = \frac{\alpha}{1-\alpha} \frac{MPL}{MPK_j}. \tag{7}$$

Also, we can write output as

$$y_j = A_j k_j^{a} l_j^{1-\alpha} = y_j = A_j \left(\frac{k}{l_j}\right)^{a} l_j = A_j \left(\frac{l_j}{k_j}\right)^{1-\alpha} k_j \tag{8}$$

Combining (7) and (8), we have:

$$k_j = \frac{y_j}{A_j} \left(\frac{k_j}{l_j}\right)^{1-\alpha} = \frac{y_j}{A_j} \left(\frac{\alpha}{1-\alpha} \frac{MPL}{MPK_j}\right)^{1-\alpha}; \quad l_j = \frac{y_j}{A_j} \left(\frac{\alpha}{1-\alpha} \frac{MPL}{MPK_j}\right)^{-\alpha}. \tag{9}$$

Using the demand function, $y_j = p_j^{-\eta} Y$ to replace y_j, we can now express k_j and l_j as linear *functions Y:*

$$k_j = \frac{p_j^{-\eta}}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{1-\alpha} Y; \quad l_j = \frac{p_j^{-\eta}}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{-\alpha} Y.$$

Using the resource constraint to integrate k_j and l_j across all firms:

$$\begin{aligned} \int k_j dj &= \int \frac{p_j^{-\eta}}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{1-\alpha} dj \cdot Y = K \\ \int l_j dj &= \int \frac{p_j^{-\eta}}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{-\alpha} dj \cdot Y = L. \end{aligned}$$

We can therefore express $K^\alpha L^{1-\alpha}$ as a linear function of Y to write:

$$\begin{aligned} Y &= \frac{1}{\left[\int \frac{p_j^{-\eta}}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{1-\alpha} dj\right]^{\alpha}\left[\int \frac{p_j^{-\eta}}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{-\alpha} dj\right]^{1-\alpha}} K^\alpha L^{1-\alpha} \\ & \frac{1}{\left[\int \frac{p_j^{-\eta}}{A_j}\left(\frac{1}{(MPK_j/R)}\right)^{1-\alpha} dj\right]^{\alpha}\left[\int \frac{p_j^{-\eta}}{A_j}\left(\frac{1}{(MPK_j/R)}\right)^{-\alpha} dj\right]^{1-\alpha}} K^\alpha L^{1-\alpha}. \end{aligned} \tag{10}$$

Note that the term can be interpreted as effective TFP. Our goal is to express this as a function of primitive productivity and distortions. To this end, we need to express p_j as a function of the wedges.

Using the factor shares equation (5) and (6), we have:

$$(1-\tau_j)\left(1-\frac{1}{\eta}\right)p_j y_j = MPK_j k_j + MPL l_j. \tag{11}$$

Note that equations (9) expresses k_j and l_j as linear functions of y_j. Using this to replace k_j and l_j in (11), we write the RHS of the above equation as:

$$\begin{aligned} MPK_j k_j + MPL l_j &= MPK_j \frac{y_j}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{1-\alpha} + MPL\frac{y_j}{A_j}\left(\frac{\alpha}{1-\alpha}\frac{MPL}{MPK_j}\right)^{-\alpha} \\ &= \frac{y_j}{A_j}\left(\frac{MPL}{1-\alpha}\right)^{1-\alpha}\left(\frac{MPK_j}{\alpha}\right)^{\alpha} \end{aligned} \tag{12}$$

Combining (11) and (12) and eliminate y_j on both sides, we have:

$$p_j = \frac{1}{(1-\tau_j)\left(1-\frac{1}{\eta}\right)A_j}\left(\frac{MPL}{1-\alpha}\right)^{1-\alpha}\left(\frac{MPK_j}{\alpha}\right)^{\alpha}. \tag{13}$$

We have now expressed p_j as a function of the productivities and prices. We can simplify this. We use the normalization condition for prices, $P = \left[\int p_j^{1-\eta} dj\right]^{\frac{1}{1-\eta}} = 1$, to integrate (13) across all j to get:

$$\left\{\int \left[\frac{1}{\left(1-\frac{1}{\eta}\right)(1-\tau_j) A_j}\left(\frac{MPL}{1-\alpha}\right)^{1-\alpha}\left(\frac{MPK_j}{\alpha}\right)^{\alpha}\right]^{1-\eta} dj\right\}^{\frac{1}{1-\eta}} = 1.$$

Therefore,

$$\frac{1}{\left(1-\frac{1}{\eta}\right)}\left(\frac{MPL}{1-\alpha}\right)^{1-\alpha}\left(\frac{R}{\alpha}\right)^{\alpha}\left\{\int \left[\frac{1}{(1-\tau_j) A_j}\left(\frac{MPK_j}{R}\right)^{\alpha}\right]^{1-\eta} dj\right\}^{\frac{1}{1-\eta}} = 1 \tag{14}$$

Use (14) to replace the economy-wide prices in (13), we have:[1]

$$p_j = \frac{\frac{1}{(1-\tau_j)A_j}\left(\frac{MPK_j}{R}\right)^{\alpha}}{\left\{\int \left[\frac{1}{(1-\tau_j)A_j}\left(\frac{MPK_j}{R}\right)^{\alpha}\right]^{1-\eta} dj\right\}^{\frac{1}{1-\eta}}}.$$

Now we substitute prices in equation (10) to get

$$Y = \frac{\left\{\int \left[\frac{1}{(1-\tau_j)A_j}\left(\frac{MPK_j}{R}\right)^{\alpha}\right]^{1-\eta} dj\right\}^{\frac{\eta}{\eta-1}}}{\left[\int \frac{\left[\frac{1}{(1-\tau_j)A_j}\left(\frac{MPK_j}{R}\right)^{\alpha}\right]^{-\eta}}{A_j}\left(\frac{1}{(MPK_j/R)}\right)^{1-\alpha} dj\right]^{\alpha}\left[\int \frac{\left[\frac{1}{(1-\tau_j)A_j}\left(\frac{MPK_j}{R}\right)^{\alpha}\right]^{-\eta}}{A_j}\left(\frac{1}{(MPK_j/R)}\right)^{-\alpha} dj\right]^{1-\alpha}} K^{\alpha} L^{1-\alpha},$$

and simplify terms to obtain (3). ■

Remark 1 1. *In the absence of any distortions, equation (3) implies:*

$$\widehat{TFP} = \left\{\int A_j^{\eta-1} dj\right\}^{\frac{1}{\eta-1}} = \widehat{TFP}\left\{\int a_j^{\eta-1} dj\right\}^{\frac{1}{\eta-1}}, \tag{15}$$

where $\widehat{TFP}$ denotes the first best level of TFP and $a_j = \frac{A_j}{\widehat{TFP}}$ is the normalized productivity. Clearly, $\left\{\int a_j^{\eta-1} dj\right\}^{\frac{1}{\eta-1}} = 1$. Under this normalization, $\left\{\int a_j^{\eta-1} dj\right\}^{\frac{1}{\eta-1}}$ can be interpreted as the efficiency margin. It equals one in the first best case.

[1] In Hsieh and Klenow's setup, $\frac{MPK_j}{R} = 1 + \tau_{K,j}$. Therefore this procedure eliminated all prices in the equation for prices. We can also set $R = 1$. In this case, our procedure represent effective TFP as a function of the marginal product of capital.

2. In the absence of output distortions (capital distortion only), we have:

$$TFP = \frac{\left\{\int \left[\frac{A_j}{(MPK_j/R)^{\alpha}}\right]^{\eta-1} dj\right\}^{\frac{1+\alpha\eta-\alpha}{\eta-1}}}{\left[\int \frac{A_j^{\eta-1}}{(MPK_j/R)^{1+\alpha\eta-\alpha}} dj\right]^{\alpha}} \tag{16}$$

The Case of Perfect Competition The Hsieh and Kelnow setup assumes monopolistic competition. The monopolistic competition setup is unnecessary. Several remarks are in order.

- To measure misallocation, it is important to take into account of the non-trivial elasticity of substitution across varieties (why?). Therefore, the Dixit-Sitglitz production function is essential.
- The market power part is inessential. In fact, in the static setup, market power itself does not distort allocations. Market power means that production scale is below its socially optimal level and the marginal product of factors are above its price (marginal cost). However, in the CES case, markup is constant. If all resources are exhausted, like in the static setup of Hsieh and Klenow, market power itself does not create misallocation of resource in the case of CES.
- In the dynamic setup, market power can distort the intertemporal margin. Unless that is what we are interested, there is no need to model market power.

We can redo the exercise above by assuming perfect competition. In this case, the intermediate goods producer's problem beomes

$$\begin{aligned} \pi(j) &= \max_{k_j, l_j} \{(1-\tau_j)\, p_j y_j - MPK_j k_j - MPL \times l_j\} \qquad (17) \\ subject\ to &: \ y_j = A_j k_j^a l_j^{1-\alpha}. \end{aligned}$$

The first order condition with respect to k_j is:

$$(1-\tau_j)\,\alpha p_j A_j k_j^{a-1} l_j^{1-\alpha} = MPK_j. \tag{18}$$

Using the fact $p_j = \left[\frac{y_j}{Y}\right]^{-\frac{1}{\eta}}$, we can see equation (18) only differs (4) by the blue colored term $\left(1-\frac{1}{\eta}\right)$. In fact, if we delete the blue-colored terms in the proof of Lemma 1, we obtain the result for the perfect competition case. That is to say, Lemma 1 continue to hold (except that we need to delete the blue-colored terms) under perfect competition.

In the case of monopolistic competition, $\frac{1}{\eta}$ is the markup. It is the fraction of total revenue that goes into the monopolistic rent. The ratio of the rental price of capital (labor) is $\left(1-\frac{1}{\eta}\right)$ fraction of the marginal product of capital (labor). Therefore, in the case of monopolistic competition, labeling the rental price of capital as MPK and the wage rate of labor as MPL are misleading, as they are not equal. In the case of perfect competition, of course, they are appropriate.

Measurement Hsieh and Klenow write $MPK_j = (1+\tau_{K,i})\,MPK$ and think of $(1+\tau_{K,i})$ as price distortions. The exercise in Hsieh and Klenow is to measure the distortions from the data and ask the welfare consequence of these distortions. The main measurement issue is that $(1-\tau_{K,i})$, $(1-\tau_i)$ and the physical productivity A_i are not observable. The structual model links them to observable quantities.

- *Capital distortion.*

  From equations (5) and (6), we have:

$$\frac{\alpha}{1-\alpha} = \frac{(1+\tau_{K,i})\,MPK}{MPL}\frac{k_j}{l_j},$$

  that is,

$$(1+\tau_{K,i}) = \frac{\alpha}{1-\alpha}\frac{MPL \times l_j}{MPK \times k_j}.$$

  $MPKL$ is the economy wide wage rate and MPK is the economy wide capital rental rate. Therefore, $(1+\tau_{K,i})$ can be identified from the relative share of capital with respect to labor in the data.

- *Labor distortion*

  Similarly, from the factor share equation, we have

$$(1-\tau_j) = \frac{MPL \times l_j}{(1-\alpha)\left(1-\frac{1}{\eta}\right)p_j y_j},$$

  That is, output distortion can be identified from labor share and mark-up.

- *Productivity*

  To account for the benefit of reallocation, one also needs to know $A_j = \frac{y_j}{k_j^{\alpha} l_j^{1-\alpha}}$. However, y_j is not directly observable. However, $p_j y_j$ is. Using the demand function, $p_j y_j = y_j^{1-\frac{1}{\eta}} Y^{\frac{1}{\eta}}$. Hence, $y_j = \left[\frac{p_j y_j}{Y^{1/\eta}}\right]^{\frac{1}{1-\frac{1}{\eta}}}$. Therefore, $A_j = Y^{-\frac{1}{\eta-1}} \times \frac{(p_j y_j)^{\frac{\eta}{\eta-1}}}{k_j^{\alpha} l_j^{1-\alpha}}$.

- *TFPQ and TFPR*

  This literature uses the terminology of TFPQ and TFPR.

$$TFPQ = A_j = \frac{y_j}{k_j^{\alpha} l_j^{1-\alpha}}$$

does not equalize across firms. However, $TFPR$ is defined as $\frac{p_j y_j}{k_j^\alpha l_j^{1-\alpha}}$ and has to equalize across all firms. To see this, by (13),

$$\begin{aligned}\frac{p_j y_j}{k_j^\alpha l_j^{1-\alpha}} &= \frac{y_j}{(1-\tau_j)\left(1-\frac{1}{\eta}\right)A_j k_j^\alpha l_j^{1-\alpha}}\left(\frac{MPL}{1-\alpha}\right)^{1-\alpha}\left(\frac{MPK_j}{\alpha}\right)^{\alpha} \\ &= \frac{1}{(1-\tau_j)\left(1-\frac{1}{\eta}\right)}\left(\frac{MPL}{1-\alpha}\right)^{1-\alpha}\left(\frac{MPK_j}{\alpha}\right)^{\alpha},\end{aligned}$$

must be equal across firms in the absence of distortions. TFPR is essentially a compensive measure of MPK and MPL across firms.

II Aggregation

To focus on capital misallocation, let's first assume that there is no labor misallocation. Structual models will generate a joint distribution of (A_i, k_i), and we assume that the allocation of labor is chosen optimally given the distribution of (A_i, k_i). Given the assumption on the production function, our main goal here is to compute 1) the operating profit of a firm with the state variable (A, k_i), and 2) the capital misallocation (EF measure) for a given distribution (A_i, k_i). We first discuss the case of CES production funtion and then the case of perfect substitutes.

Dixit-Stiglitz production function We focus on the case of perfect competition. The case of monopolistic competition can be easily derived based on the insights from the last section. We think of (A_i, k_i) as the state variable of a firm. In the background, there might be some financial market frictions that distort the allocation of k_i. However, labor market is perfectly efficient, and l_i is chose optimally on a competitive market in each firm.

In this case, firm j 's profit maximization problem is

$$\begin{aligned}TR(A_j, k_j) &= \max_{l_j}\{p_j y_j - MPL \times l_j\} \\ \textit{subject to} &: \; y_j = A_j k_j^a l_j^{1-\alpha}.\end{aligned} \tag{19}$$

The market clearing condition $\int l_i di = L$ determine the wage rate MPL. We are interested in understanding the functional form of $TR(A_j, k_j)$ and the implied misallocation. We summarize our result in the following lemma.

Lemma 2 *Assume that there are no output distortions. In the case of perfect competition,* *1. The total revenue and the marginal product of capital in firm j satisfies:*

$$TR_j = \alpha\left\{\int A_j^{(\eta-1)(1-\xi)}k_j^{\xi}dj\right\}^{\frac{1}{(\eta-1)(1-\xi)}-1} L^{1-\alpha}\times A_j^{(\eta-1)(1-\xi)}k_j^{\xi} = \alpha p_j y_j, \quad (20)$$

$$MPK_j = \alpha\left\{\int A_j^{(\eta-1)(1-\xi)}k_j^{\xi}dj\right\}^{\frac{1}{(\eta-1)(1-\xi)}-1} L^{1-\alpha}\times A_j^{(\eta-1)(1-\xi)}k_j^{\xi-1} = \frac{TR_j}{k_j}. \quad (21)$$

2. The implied total output and misallocation are given by:

$$Y = \left\{\int A_j^{(\eta-1)(1-\xi)}k_j^{\xi}dj\right\}^{\frac{1}{(\eta-1)(1-\xi)}} L^{1-\alpha}; \quad EF = \left\{\frac{\int A_j^{(\eta-1)(1-\xi)}k_j^{\xi}dj}{\left[\int A_j^{\eta-1}dj\right]^{1-\xi}K^{\xi}}\right\}^{\frac{1}{(\eta-1)(1-\xi)}}. \quad (22)$$

Proof. *We first solve for the equilibrium allocation of labor. Because labor share must be $1-\alpha$, we must have (In fact, (23) is also the first order condition on l_j):*

$$(1-\alpha)\,p_j y_j = MPL l_j. \quad (23)$$

In equilibrium, $p_j = y_j^{-\frac{1}{\eta}}Y^{\frac{1}{\eta}}$; therefore,

$$p_j y_j = y_j^{1-\frac{1}{\eta}}Y^{\frac{1}{\eta}} = \left[A_j k_j^{a} l_j^{1-\alpha}\right]^{1-\frac{1}{\eta}}Y^{\frac{1}{\eta}}. \quad (24)$$

Equation (23) can therefore be written as:

$$(1-\alpha)\left[A_j k_j^{a}\right]^{1-\frac{1}{\eta}}Y^{\frac{1}{\eta}} = MPL l_j^{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)},$$

which implies $l_j \propto \left[A_j k_j^{a}\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}$. Using the resource constraint, $\int l_j = L$, we have:

$$l_j = \frac{\left[A_j k_j^{a}\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}}{\int\left[\left[A_j k_j^{a}\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}\right]dj}L.$$

Next, using the equilibrium allocation of labor, we can derive firms' (capital) revenue function.

The total revenue is

$$p_j y_j = \left[A_j k_j^a l_j^{1-\alpha}\right]^{1-\frac{1}{\eta}} Y^{\frac{1}{\eta}} = \left[A_j k_j^a \left\{\frac{\left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}} L}{\int\left[\left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}\right] dj}\right\}^{1-\alpha}\right]^{1-\frac{1}{\eta}} Y^{\frac{1}{\eta}}$$

$$= \frac{\left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}} L^{(1-\alpha)\left(1-\frac{1}{\eta}\right)}}{\left\{\int\left[\left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}\right] dj\right\}^{(1-\alpha)\left(1-\frac{1}{\eta}\right)}} Y^{\frac{1}{\eta}} \tag{25}$$

Note that $\int p_j y_j dj = Y$ *implies that*

$$Y = \left\{\int\left[\left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}\right] dj\right\}^{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)} L^{(1-\alpha)\left(1-\frac{1}{\eta}\right)} Y^{\frac{1}{\eta}},$$

that is, $Y = \left\{\int\left[\left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}\right] dj\right\}^{\frac{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}{1-\frac{1}{\eta}}} L^{1-\alpha}$*. Using (**??**), we have:*

$$p_j y_j = \left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}} \left\{\int\left[\left[A_j k_j^a\right]^{\frac{1-\frac{1}{\eta}}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)}}\right] dj\right\}^{\frac{1-\alpha+\alpha\eta}{\eta-1}-1} L^{1-\alpha}.$$

Denote $\xi = \frac{\alpha\left(1-\frac{1}{\eta}\right)}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)} = \frac{\alpha\eta-\alpha}{1+\alpha\eta-\alpha}$*. Note that this definition implies that* $1-\xi = \frac{1}{1+\alpha\eta-\alpha}$ *and* $\frac{\xi}{1-\xi} = \alpha(\eta-1)$*. The above can be simplified to:*

$$p_j y_j = A_j^{(\eta-1)(1-\xi)} k_j^{\xi} \left\{\int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj\right\}^{\frac{1}{(\eta-1)(1-\xi)}-1}; L^{1-\alpha} \quad Y = \left\{\int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj\right\}^{\frac{1}{(\eta-1)(1-\xi)}} L^{1-\alpha} \tag{26}$$

Perfect competition implies that capital share is α*, the profit function is therefore:*

$$TR(A_j, k_j) = \alpha A_j^{(\eta-1)(1-\xi)} k_j^{\xi} \left\{\int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj\right\}^{\frac{1}{(\eta-1)(1-\xi)}-1} L^{1-\alpha}. \tag{27}$$

This proves (20).

To compute the efficiency measure, note that Equation (27) obviously implis that the efficient allocation of capital must satisfy $k_j \propto A_j^{\eta-1}$; therefore, $k_j = \frac{A_j^{\eta-1}}{\int A_j^{\eta-1} dj} K$. Equation (26) therefore implies that first best output is

$$\hat{Y} = \left\{ \int A_j^{(\eta-1)(1-\xi)} \left[\frac{A_j^{\eta-1}}{\int A_j^{\eta-1} dj} \right]^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}} K^{\frac{\xi}{(\eta-1)(1-\xi)}} L^{1-\alpha} = \left\{ \int A_j^{(\eta-1)} dj \right\}^{\frac{1}{(\eta-1)}} K^{\alpha} L^{1-\alpha},$$

as $\frac{\xi}{(\eta-1)(1-\xi)} = \alpha$. Therefore

$$EF = \frac{Y}{\hat{Y}} = \frac{\left\{ \int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}}}{\left\{ \int A_j^{(\eta-1)} dj \right\}^{\frac{1}{(\eta-1)}} K^{\alpha}},$$

which proves (22) given the definition of ξ.

To compute the marginal product of capital, we have:

$$MPK_j = MPK_j = p_j \frac{\partial y_j}{\partial k_j} = \alpha A_j^{(\eta-1)(1-\xi)} k_j^{\xi-1} \left\{ \int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}-1}.$$

■

Note that the technology is CRS. However, equation (27) implies that profit function is DRS with respect to capital. In competitive equilibrium, given the output of all other varieties, the profit for firm j is DRS, because of the imperfect substitutability. However, we cannot compute MPK_j by taking the first-order derivative of $TR(A_j, k_j)$. The reason is that the DRS of $TR(A_j, k_j)$ with respect to k_j came from the dependence of p_j on k_j, which firms take as given. Despite the DRS of $TR(A_j, k_j)$, firms do not early any profit, because they are price takers and technology is CRS.

Perfect substitution and DRS Let's now consider the case in which technology is CRS but different varieties are perfect substitutes. We continue to assume that there are a unit measure of firms and write $Y = \int y_j dj$. Clearly, in this case, intermediate goods producers are perfectly competitive. We assume the following general functional form of the production function

$$y_i = A \left[(\theta_i k_i)^{\beta\nu} L_i^{1-\beta} \right], \tag{28}$$

which as we will see, will allow us to model both CRS and DRS.

In the case of DRS, $\nu < 1$. It is more convenient to define $z_i^{1-\nu} = \theta_i^{\nu}$, that is, $z_i = \theta_i^{\frac{\nu}{1-\nu}}$. In this case, the production function can be written as:

$$y_i = A\left[\left(z_i^{1-\nu}k_i^{\nu}\right)^{\beta} L_i^{1-\beta}\right]. \tag{29}$$

Lemma 3 *Assume that firm level output is given by (29),*

1. *The total revenue and the marginal product of capital in firm j satisfies:*

$$\begin{aligned} TR_j &= \beta A z_i^{1-\nu}k_i^{\nu}L^{1-\beta}\left(\int z_i^{1-\nu}k_i^{\nu}di\right)^{\beta-1} = \beta y_i, & (30)\\ MPK_j &= \beta\nu A z_i^{1-\nu}k_i^{\nu-1}L^{1-\beta}\left(\int z_i^{1-\nu}k_i^{\nu}di\right)^{\beta-1} = \nu\frac{TR_i}{k_i}. & (31)\end{aligned}$$

2. *The implied total output and misallocation are given by:*

$$Y = A\left(\int z_i^{1-\nu}k_i^{\nu}di\right)^{\beta}L^{1-\beta};\quad EF = \left(\frac{\int z_i^{1-\nu}k_i^{\nu}di}{\left(\int z_i di\right)^{1-\nu}K^{\nu}}\right)^{\beta}. \tag{32}$$

Proof. *Optimization on labor market implies that the profit must be proportional to $z_i^{1-\nu}k_i^{\nu}$:*

$$TR\left(z_i,k_i\right) = \max\left\{A\left[\left(z_i^{1-\nu}k_i^{\nu}\right)^{\beta}L_i^{1-\beta}\right] - wL_i\right\} = \bar{\pi}z_i^{1-\nu}k_i^{\nu}, \tag{33}$$

for some $\bar{\pi}$.

To derive the expression for $\bar{\pi}$, note that optimality implies that $L_i = \frac{z_i^{1-\nu}K_i^{\nu}}{\int z_i^{1-\nu}K_i^{\nu}di}L$, and we can write output as:

$$y_i = A\left[\left(z_i^{1-\nu}k_i^{\nu}\right)^{\beta}L_i^{1-\beta}\right] = A z_i^{1-\nu}k_i^{\nu}L^{1-\beta}\left(\int z_i^{1-\nu}k_i^{\nu}di\right)^{\beta-1}.$$

Again, labor share is $1-\beta$. Therefore, capital share+profit share is β. We therefore have:

$$TR\left(\theta_i,k_i\right) = \beta A z_i^{1-\nu}k_i^{\nu}L^{1-\beta}\left(\int z_i^{1-\nu}k_i^{\nu}di\right)^{\beta-1},$$

and

$$\bar{\pi} = \beta A L^{1-\beta}\left(\int z_i^{1-\nu}k_i^{\nu}di\right)^{\beta-1}. \tag{34}$$

To compute the marginal product of capital, we have

$$MPK_j = \frac{\partial}{\partial k_j}TR\left(\theta_i,k_i\right) = \beta\nu A z_i^{1-\nu}k_i^{\nu-1}L^{1-\beta}\left(\int z_i^{1-\nu}k_i^{\nu}di\right)^{\beta-1}.$$

Given a joint distribution of (z_i, k_i)*, total output is given by:*

$$Y = \int y_i di = A \left(\int z_i^{1-\nu} k_i^{\nu} di \right)^{\beta} L^{1-\beta}.$$

To derive an expression for the efficiency ratio, note that the convenience of our parameterization is that the optimal allocation of capital requires $k_i = \frac{z_i}{\int z_i di} K$*, where* K *is total capital stock. In this case,* $z_i^{1-\nu} k_i^{\nu} = z_i K^{\nu} \left(\int z_i di \right)^{-\nu}$ *and* $\int z_i^{1-\nu} k_i^{\nu} di = \left(\int z_i di \right)^{1-\nu} K^{\nu}$*. Therefore, total output under the frictionless case is*

$$Y^{EF} = A \left[\left(\int z_i di \right)^{1-\nu} K^{\nu} \right]^{\beta} L^{1-\beta}.$$

The efficiency of capital reallocation can be measured by:

$$EF = \left(\frac{\int z_i^{1-\nu} k_i^{\nu} di}{\left(\int z_i di \right)^{1-\nu} K^{\nu}} \right)^{\beta}.$$

■

Compare Lemma 2 with Lemma 3, if we set $z_i = A_i^{\eta-1}$, and $\nu = \xi$, and $\beta = \frac{1}{(\eta-1)(1-\xi)}$ then in both cases,

$$TR_j \propto z_i^{1-\nu} k_i^{\nu} \left[\int z_i^{1-\nu} k_i^{\nu} di \right]^{\beta-1}; \quad EF = \left(\frac{\int z_i^{1-\nu} k_i^{\nu} di}{\left(\int z_i di \right)^{1-\nu} K^{\nu}} \right)^{\beta}.$$

Also note that in the case of CRS Dixit-Stiglitz technology, $\frac{TR_j}{p_j y_j} = \alpha < \frac{1}{(\eta-1)(1-\xi)}$, and in the case of DRS Cobb-Douglas technology, $\frac{TR_j}{y_j} = \beta$. Therefore, we can choose the parameters of the DS technology and the Cobb Douglas technology so that misallocation are the same, but the proft share of the Cobb-Douglas technology is higher.

Perfect substitution and CRS The general formulation (28) allows us to consider CRS by setting $\nu = 1$. In this case, the production function can be written as $y_i = A \left[(\theta_i K_i)^{\alpha} L_i^{1-\alpha} \right]$.

Lemma 4 *Assume that the production function is given by* $y_i = A \left[(\theta_i K_i)^{\alpha} L_i^{1-\alpha} \right]$*, where* $\theta \in [\theta_L, \theta_H]$.

1. *The total revenue and the marginal product of capital in firm* j *satisfies:*

$$TR_j = \alpha A \theta k_j \left(\int \theta_j k_j dj \right)^{\alpha-1} L^{1-\alpha} = \alpha y_j, \quad (35)$$

$$MPK_j = \alpha A \theta k_j \left(\int \theta_j k_j dj \right)^{\alpha-1} L^{1-\alpha} = \frac{TR_j}{k_j}. \quad (36)$$

2. *The implied total output and misallocation are given by:*

$$Y = A\left(\int z_i^{1-\nu} k_i^{\nu} di\right)^{\beta} L^{1-\beta}; \quad EF = \left(\frac{\int z_i^{1-\nu} k_i^{\nu} di}{\left(\int z_i di\right)^{1-\nu} K^{\nu}}\right)^{\beta}. \tag{37}$$

Optimization on labor market implies that the revenue for a firm with (θ_i, K_i) is:

$$Rev\left(\theta_i, K_i\right) = \max_{L_i}\left\{A\left(\theta_i K_i\right)^{\alpha} L_i^{1-\alpha} - wL_i\right\} = MPK\left(\theta_i\right) K_i, \tag{38}$$

where the last equality is due to CRS. Assuming total labor supply equals L, we must have $L_i = \frac{\theta_i K_i}{\int \theta_i K_i di} L$. Therefore,

$$y_i = A\left(\theta_i K_i\right)^{\alpha} L_i^{1-\alpha} = A\theta_i K_i \left(\int \theta_i K_i di\right)^{\alpha-1} L^{1-\alpha}.$$

Because labor share is $1-\alpha$, we must have:

$$MPK\left(\theta_i\right) K_i = \alpha y_i = \alpha A \theta_i K_i \left(\int \theta_i K_i di\right)^{\alpha-1} L^{1-\alpha},$$

which implies:

$$MPK\left(\theta\right) = \alpha A \theta \left(\int \theta_i K_i di\right)^{\alpha-1} L^{1-\alpha}. \tag{39}$$

It is also clear that given a joint distribution of (θ_i, K_i), total output is given by:

$$Y = \int y_i di = A\left(\int \theta_i K_i di\right)^{\alpha} L^{1-\alpha}.$$

Obviously, optimal allocation of capital requires that capital goes to the most efficient unit. Therefore the output under efficient allocation is $Y^{EF} = A\left(\theta_H K\right)^{\alpha} L^{1-\alpha}$. The efficiency of capital reallocation can be measured by:

$$EF = \left(\frac{\int \theta_i K_i di}{z_H \int K_i di}\right)^{\alpha}.$$

III Learning

Suppose there is measure one of firms, and suppose $\ln A_j$ is log-normally distributed. Consider two cases:

1. Allocation, K_j and N_j are chosen after A_j is observed. In this case, (15) implies that $Y =$

$\widehat{\mathbf{A}}K^{\alpha}L^{1-\alpha}$, where

$$\begin{aligned}\widehat{\mathbf{A}} &= \left\{\int A_j^{\eta-1}dj\right\}^{\frac{1}{\eta-1}} = \left\{\int e^{(\eta-1)\ln A_j}dj\right\}^{\frac{1}{\eta-1}} \\ &= \left\{e^{(\eta-1)\mu+\frac{1}{2}(\eta-1)^2\sigma^2}\right\}^{\frac{1}{\eta-1}} \\ &= e^{\mu+\frac{1}{2}(\eta-1)\sigma^2}\end{aligned}$$

2. Allocatioins are chosen before observing A_j. All firms are ex ante identical, capital and labor must be allocated equally across firms. In this case,

$$Y = \left\{\int \left[A_j K^{\alpha}L^{1-\alpha}\right]^{\frac{\eta-1}{\eta}} dj\right\}^{\frac{\eta}{\eta-1}} = \left[\int A_j^{\frac{\eta-1}{\eta}} dj\right]^{\frac{\eta}{\eta-1}} K^{\alpha}L^{1-\alpha}.$$

Therefore, $Y = \overline{\mathbf{A}}K^{\alpha}L^{1-\alpha}$, where

$$\overline{\mathbf{A}}=e^{\mu+\frac{1}{2}\frac{1}{\eta}(\eta-1)\sigma^2}.$$

Note that $\ln\frac{\widehat{\mathbf{A}}}{\overline{\mathbf{A}}} = \frac{1}{2}\sigma^2\left(1-\frac{1}{\eta}\right)(\eta-1) = \frac{1}{2}\sigma^2\frac{1}{\eta}(\eta-1)^2$.

3. In general, assume that we observe a signal $s = \ln A + e$, where $e \sim N\left(0,\tau^2\right)$. Allocations are chosen based on the signal. In this case,

$$\left\{\int \left[A_j K(s)^{\alpha} L(s)^{1-\alpha}\right]^{\frac{\eta-1}{\eta}} dj\right\}^{\frac{\eta}{\eta-1}}.$$

Note that

$$\begin{aligned}\int \left[A_j K(s)^{\alpha} L(s)^{1-\alpha}\right]^{\frac{\eta-1}{\eta}} dj &= \int E\left[e^{\frac{\eta-1}{\eta}\ln A_j}\middle| s\right]\left[K(s)^{\alpha} L(s)^{1-\alpha}\right]^{\frac{\eta-1}{\eta}} dj \\ &= \int e^{\frac{\eta-1}{\eta}E\left[\ln A_j\middle|s\right]+\frac{1}{2}\left(\frac{\eta-1}{\eta}\right)^2 Var\left[\ln A_j\middle|s\right]}\left[K(s)^{\alpha} L(s)^{1-\alpha}\right]^{\frac{\eta-1}{\eta}} dj \\ &= ?\end{aligned}$$

Some observations here. $E\left[\ln A|\, s\right] = \frac{1}{\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\left[\frac{1}{\sigma^2}\mu+\frac{1}{\tau^2}s\right]$, and $Var\left[\ln A|\, s\right] = \frac{1}{\frac{1}{\sigma^2}+\frac{1}{\tau^2}} = \frac{\tau^2}{\sigma^2+\tau^2}\sigma^2$. Note that $Var\left[\ln A\right] = E\left(Var\left[\ln A|\, s\right]\right)+Var\left(E\left[\ln A|\, s\right]\right)$. Therefore, $Var\left(E\left[\ln A|\, s\right]\right) = \sigma^2 - \frac{\tau^2}{\sigma^2+\tau^2}\sigma^2 = \frac{\sigma^2}{\sigma^2+\tau^2}\sigma^2$. We can also verify this by calculating $Var\left\{\frac{1}{\frac{1}{\sigma^2}+\frac{1}{\tau^2}}\left[\frac{1}{\sigma^2}\mu+\frac{1}{\tau^2}s\right]\right\}$ directly.