Lecture 2: Two-Period GK Model and Its Variations

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Abstract

I describe some simple variations of the GK model. First, I want to clarify some subtle timing issue in the model. Second, I will use these simple setups to understand some of the basic tradeoffs in models with financial constraints. Finally, I will introduce capital misallocation to prepare for the fully dynamic model with financial constraints and capital misallocation.

I The General Setup

Consider a two-period economy. The household preference is given by:

$$\ln C_0 + \beta E\left[\ln C_1\right].$$

Household owns an initial wealth W_0 and shares in the bank. The household can only invest in a risk-free assset, the return on which is denoted R_f. Therefore, the household's utility maximization problem can be written as:

$$(HH): \max_{C_0, C_1, B_f} \left\{\ln C_0 + \beta \ln C_1\right\} \tag{1}$$

$$C_0 + B_f = W_0 \tag{2}$$

$$C_1 = B_f R_f + n_1. \tag{3}$$

Here, I use B_f to denote the total savings of the household in the risk-free bond. I use n_1 to denote the profit from the bank. Note that the market is incomplete. In particular, the household cannot reallocate C_1 across states of the world freely.

Capital accumulation firms take the price of capital as given and produce capital goods:

$$\begin{aligned} (FM) \quad &: \quad \max\left\{Q_1 K_1 - H\left(I, K_0\right)\right\} \qquad (4) \\ K_1 \quad &= \quad (1-\delta) K_0 + I, \end{aligned}$$

where $H\left(, K\right)$ is adjustment cost. This implies that $Q_1 = H_I\left(I, K_0\right)$, whenever the firm is free to invest. Also, the resource constraint is given by:

$$C_0 + H\left(I, K_0\right) = A_0 K_0, \tag{5}$$

$$C_1 = A_1 K_1, \tag{6}$$

$$K_1 = (1-\delta) K_0 + I \tag{7}$$

where A_0 and A_1 are productivity shocks.

For simplicity, we make several assumptions on the parameters to facilitate computation. We assume $\beta = 1$, and $H(I, K) = I$. That is, there is no adjustment cost. As a result, the price of capital equals one whenever the firm is free to invest. Finally, we assume $E[A^{-1}] = 1$ for simplicity.

II One-Sector, Ex Post Default

I first describe a one-sector GK model with expost default.

1. The household problem. The HH takes his initial wealth inital wealth W_0, interest rate R_f, and bank profit n_1 as given and choose $\hat{C}_0$, $\hat{C}_1$, $\hat{B}_f$ to solve the utility maximization problem in (1)-(3).

2. There is only one type of firm. The firm takes the price of capital as given and solve the profit maximiztion problem as in (4).

   Because the HH and FM problem are the same in all variations of the the model. We will not repeat the HH and FM's problem, and focus on the bank's problem.

3. The Bank's problem. The bank chooses $\hat{K}_1$, $\hat{B}_f$, $\hat{n}_1$ to maximize profit, given initial wealth n_0, interest rate R_f, and price of capital, Q_1:

$$(BK) : \max \{E[Mn_1]\} \tag{8}$$

$$n_1 = A_1 K_1 - R_f B_f \tag{9}$$

$$n_1 \geq 0 \tag{10}$$

$$Q_1 K_1 = n_0 + B_f \tag{11}$$

   Note that the constraint $n_1 \geq 0$ captures that the bank can default. We call this expost default, because the bank can choose to default as the realization of the productivity shock. Our notion of equilibrium requires that the bank does not have the incentive to default *in all possible states* of the world.

Given the initial endowment for the household and the bank $\{W_0, n_0\}$, equilibrium consists of i) prices, R_f, Q_0, Q_1 and bank profit n_1; ii) Allocations $\hat{C}_0$, $\hat{C}_1$, $\hat{B}_f$, $\hat{I}_1$, $\hat{K}_1$, such that

1. Given W_0, R_f and n_1, $\left\{\hat{C}_0, \hat{C}_1, \hat{B}_f\right\}$ solve the HH programming problem, (HH).

2. Given n_0, R_f and Q_1, $\left\{\hat{K}_1, \hat{n}_1, \hat{B}_f\right\}$ solves the bank's programming problem, (BK).

3. Given Q_0 and Q_1, $\left\{\hat{I}, \hat{K}_1\right\}$ solves the firm's programming problem, (FM).

4. In addition, the following market clearing conditions (5)-(7) are satisfied.

A couple of remarks:

- I do not include firm's profit, as CRS implies that firm does not make any profit in equilibrium as in neoclassical models.

- Note that n_1 and C_1 are functions functions of the realizations of shocks, A_1.

A Equilibrium Conditions

Note that the (HH) implies

$$\beta E\left[\frac{C_0}{C_1}\right] R_f = 1, \tag{12}$$

Using the resource constraint, $C_0 = A_0 K_0 - H(I, K_0)$, $C_1 = A_1 [(1-\delta) K_0 + I]$. Therefore,

$$R_f = \frac{1}{\beta E\left[\frac{A_0 K_0 - H(I,K_0)}{A_1[(1-\delta)K_0+I]}\right]} = \frac{1}{\beta E\left[A_1^{-1}\right]} \frac{(1-\delta) K_0 + I}{[A_0 K_0 - H(I, K_0)]}. \tag{13}$$

Under our assumption of the parameter value,

$$R_f = \frac{(1-\delta) K_0 + I}{A_0 K_0 - I}.$$

Consider the optimization problem of FI. We can write the objective function as:

$$\max\left\{E\left[M\left\{A_1 \frac{n_0 + B_f}{Q_1} - R_f B_f\right\}\right]\right\}.$$

First order condition with respect to B_f implies

$$\frac{E[MA_1]}{Q_1} \geq E[M_1]R_f, \quad " = " \; holds \; if \; n_1 > 0 \; (for \; all \; realizations \; of \; A_1),$$

where

$$M = \frac{\beta C_0}{C_1} = \frac{\beta [A_0 K_0 - I]}{A_1 [(1-\delta) K_0 + I]}.$$

Combine with (12), we have:

$$\frac{\beta [A_0 K_0 - I]}{[(1-\delta) K_0 + I]} \geq 1; \; = \; if \; not \; binding. \tag{14}$$

B Case 1: Frictionless Benchmark

We ignore the GK constraint and solve the equilibruim. In this case, (14) and (**??**) jointly implies

$$1 = \frac{\beta [A_0 K_0 - I]}{[(1-\delta) K_0 + I]} \tag{15}$$

which completely determines optimal investment:

$$I^* = \frac{1}{2} [A_0 - (1-\delta)] K_0$$

Given (15), the equilibrium quantity and prices can be constructed as:

$$\begin{aligned} C_0^* &= A_0 K_0 - I^*; \\ K_1^* &= (1-\delta) K_0 + I^*; \\ C_1^* &= A_1 K_1^*; \end{aligned}$$

and prices are given by:

$$\begin{aligned} R_f &= 1 \\ Q_0 &= A_0 + (1-\delta) \end{aligned}$$

For this to be a valid equilibrium, we need:

$$A_{\min} K_1^* \geq R_f^* \left[Q_1^* K_1^* - n_0 \right], \tag{16}$$

which is a constraint on the bank's net worth, n_0. Under our parameter values, this implies that the minimum net worth of the finanical sector such that the above can be an equilibrium is given by:

$$\begin{aligned} n^* &= (Q_1 - A_{\min}) \left[(1-\delta) K_0 + I^* \right] \quad (17) \\ &= (1 - A_{\min}) \frac{1}{2} (A_0 + 1 - \delta) K_0. \quad (18) \end{aligned}$$

C Case 2: The GK equilibrium

In ths case (16) is violated, the frictionless equilibruim is no longer feasible. Since the constraint $n_1 \geq 0$ is binding for lowest realization of A_1, constraints (9) and (10) can imply

$$A_{\min} K_1 = R_f B_f. \tag{19}$$

Note

$$K_1 = (1-\delta) K_0 + I,$$

and by (11),

$$B_f = Q_1 K_1 - n_0 = \left[(1-\delta) K_0 + I \right] - n_0.$$

Also, by (13),

$$R_f = \frac{(1-\delta) K_0 + I}{\left[A_0 K_0 - I \right]}$$

Plug the above equations into (19), we found an equation that determines the equilibrium investment in the constrained case. The equilibrium investment $\hat{I}$ is determined by the constraint (16):

$$A_{\min} \left[A_0 K_0 - I \right] = \left\{ \left[(1-\delta) K_0 + \hat{I} \right] - n_0 \right\}. \tag{20}$$

This implies:

1. Investment is lower:

$$\hat{I} = \frac{[A_{\min} - 1 + \delta]\, K_0 + n_0}{1 + A_{\min}} < I^*,$$

   because $n_0 < n_0^*$.

2. Consumption growth lower:

$$\frac{\hat{C}_1}{\hat{C}_0} = \frac{A_1 [(1-\delta) K_0 + I]}{[A_0 K_0 - I]} < \frac{A_1 [(1-\delta) K_0 + I^*]}{[A_0 K_0 - I^*]},$$

   because $\hat{I} < I^*$.

3. Stock market is strictly more attractive (stock market valuation is low):

$$\frac{E[MA_1]}{Q_1} > E[M_1]\, R_f$$

4. Real interest is lower, because consumption growth is lower:

$$R_f = \frac{1}{\beta E\left[A_1^{-1}\right]} \frac{(1-\delta) K_0 + \hat{I}}{\left[A_0 K_0 - \hat{I}\right]} = \frac{(1-\delta) K_0 + \hat{I}}{\left[A_0 K_0 - \hat{I}\right]} < 1.$$

III Two Sectors, Ex Ante Default

As in GK, we assume that there is a fraction π of islands that have investment opportunity, and the rest of the islands do not have investment opportunity. Then bank's optimization problem is given by:

$$(BK): \max_{B_f, B_B^i} \sum_{i=I,N} \pi^i E\left[M n_1^i\right]$$

$$n_1^i = A_1 K_1^i - R_f B_f - R_B B_B^i, \qquad \forall (i, A_1)$$

$$E\left[M\left\{n_1^i\right\}\right] \geq \theta \left[Q^i K^i - \omega B_B^i\right], \quad \forall i \tag{21}$$

$$Q^i K_1^i = n_0 + B_f + B_B^i, \qquad \forall i \tag{22}$$

Finally, the interbank borrowing market must clear. This implies

$$\pi B^I + (1-\pi) B^N = 0. \tag{23}$$

It is convenient to denote $B = \pi B_B^I$. Using this notation and the market clearing condition, we have:

$$B_B^I = \frac{B}{\pi}; \quad B_B^N = -\frac{B}{1-\pi}. \tag{24}$$

Several remarks on the setup of the model:

1. This setup follows exactly GK, except that I make it two period. Note $i = I$ means that the island has an investment opportunity. $i = N$ means that it does not have the investment opportunity.

2. Here is the timing of events. Bank first decide how much to borrow from the HH, B_f. This decision is made before the realization of the investment opportunity. Next, investment opportunity realizes, the banks choose the amount of interbank borrowing and lending, B_B^i, given the realization of the investment opportunity. After that the bank can choose to default or not. Note that the default decision is made before the realization of the productivity shock on date 1. On date 1, A_1 realizes and payment are made *in full amount.*

3. Because the bank can default after seeing $i = I, N$ but before the realization of A_1, the no default constraint says the bank can never have the incentive to default. Note that after default, the bank can seize $\theta \left[Q^i K^i - \omega B_B^i\right]$ amount of asset.

4. The parameter ω captures the efficiency of interbanking lending market. $\omega = 1$ is the case of fully frictionless interbank market. $\omega = 0$ corresponds to the case in which banks are no more efficient in seizing assets than HH. See GK for details.

Consistent with the notaion is GK, it is convenient to define:

$$\nu_K = E\left[MA_1\right]; \quad \nu = E\left[MR_f\right]; \quad \nu_B = E\left[MR_B\right].$$

Of course, HH first order condition implies $\nu = 1$. However, I will keep using ν just to be consistent with KG. In this case, the bank's optimization problem is written as:

$$(BK): \max_{B_f, \left\{B_B^i, K^i\right\}_{i=I,N}} \sum_{i=I,N} \pi^i \left[\nu_K K_1^i - \nu B_f - \nu_B B_B^i\right]$$

$$\nu_K K_1^i - \nu B_f - \nu_B B_B^i \geq \theta \left[Q^i K^i - \omega B_B^i\right] \tag{25}$$

$$Q^i K_1^i = n_0 + B_f + B_B^i \tag{26}$$

Before we solve the model, we make several observations. First, assume that $Q_1^I = Q_1^N = Q_1$. Then,

$$n_0 + B_f = Q_1 \left[(1 - \delta) K_0 + I\right]. \tag{27}$$

The intuition for this is simple. Multiplying (26) by π for investment islands, and by $1 - \pi$ for non-investment islands, we have:

$$\begin{aligned} \pi \left(Q_1 K_1^I - n_0 - B_f\right) &= \pi B_B^i \\ (1 - \pi) \left(Q_1 K_1^N - n_0 - B_f\right) &= (1 - \pi) B_B^i \end{aligned}$$

Adding together, and using the market clearing condition (23), we have:

$$Q_1 \left[\pi K_1^I + (1 - \pi) K_1^N\right] = n_0 + B_f.$$

Using $K_1^I = (1 - \delta) K_0 + \frac{I}{\pi}$, and $K_1^N = (1 - \delta) K_0$, we can prove (27). Intuitively, (27) is the aggregate balance sheet of the banking sector.

Given that $n_0 + B_f = Q_1 \left[(1 - \delta) K_0 + I\right]$, and $Q_1 K_1^I = Q_1 \left[(1 - \delta) K_0 + \frac{I}{\pi}\right]$, we can derive the following expressions for the interbank loans:

$$B^N = -Q_1 I, \quad B^I = \frac{1 - \pi}{\pi} Q_1 I.$$

A First best

In the case of first best, the allocation is as characterized before:

$$\begin{aligned} C_0^* &= A_0 K_0 - I^*; \\ K_1^* &= (1-\delta) K_0 + I^*; \\ C_1^* &= A_1 K_1^*; \end{aligned}$$

and prices are given by:

$$\begin{aligned} R_f &= 1 \\ Q_0 &= A_0 + (1-\delta); \quad Q_1 = 1, \\ \nu_K &= \nu = \nu_B = 1. \end{aligned}$$

The limited commit constraint requires that for island I,

$$(1-\theta)\left[(1-\delta) K_0 + \frac{I}{\pi}\right] - B_f \geq (1-\theta) \frac{1-\pi}{\pi} I$$

and for island N:

$$(1-\theta)(1-\delta) K_0 - B_f \geq -(1-\theta) \hat{I}.$$

Note that both equations are equivalent, which boils down to

$$B_f \leq (1-\theta)\left[(1-\delta) K_0 + \hat{I}\right].$$

Combine with equation (27), we have

$$n_0 \geq \theta\left[(1-\delta) K_0 + \frac{I}{\pi}\right],$$

which is the following lemma:

Lemma 1 *(First Best)*

Let

$$n_0^* = \theta\left[(1-\delta) K_0 + \frac{I^*}{\pi}\right],$$

where $I^* = \frac{1}{2}\left[A_0 - (1-\delta)\right] K_0$ *is the first best level of investment. The first best allocation achieves if and only* $n_0 \geq n_0^*$.

B Limited commitment

Lemma 2 *(FOC)*

The first order optimality conditions imply:

$$\frac{\nu_K}{Q^i} - \nu_B = \frac{\lambda_i}{1+\lambda_i}\theta\left(1-\omega\right), \ \forall i; \tag{28}$$

$$\nu_B - \nu = \frac{\bar{\lambda}}{1+\bar{\lambda}}\theta\omega. \tag{29}$$

To derive optimality conditions in the constrained case, we set up the Lagrangian of the maximization problem, (22):

$$\sum_{i=H,L}\pi_i\left\{\nu_K\frac{n_0+B_f+B_B^i}{Q^i} - \nu B_f - \nu_B B_B^i + \lambda_i\left[\left(\nu_K - \theta Q^i\right)\frac{n_0+B_f+B_B^i}{Q^i} - \nu B_f - \left(\nu_B - \theta\omega\right)B_B^i\right]\right\}.$$

The first order condition with repsect to B_B^i implies:

$$\frac{\nu_K}{Q^i} - \nu_B + \lambda_i\left[\frac{\nu_K}{Q^i} - \nu_B - \theta\left(1-\omega\right)\right] = 0,$$

which is (28).

Note that equation (28) can be written as:

$$\left(1+\lambda_i\right)\left[\frac{\nu_K}{Q^i} - \nu_B\right] = \lambda_i\theta\left(1-\omega\right)$$

Taking expectation on both sides of the equation, we have:

$$\sum_{i=H,L}\pi_i\left(1+\lambda_i\right)\frac{\nu_K}{Q^i} = \left(1+\bar{\lambda}\right)\nu_B + \bar{\lambda}\theta\left(1-\omega\right), \tag{30}$$

where $\bar{\lambda} = E\left[\lambda\right] = \sum_{i=H,L}\pi_i\lambda_i$.

FOC with respect to B_f gives

$$\sum_{i=H,L} \pi_i \left\{ \frac{\nu_K}{Q^i} - \nu + \lambda_i \left[\frac{\nu_K}{Q^i} - \theta - \nu \right] \right\} = 0,$$

or

$$\sum_{i=H,L} \pi_i \left\{ (1+\lambda_i) \frac{\nu_K}{Q^i} \right\} - \nu \sum \pi_i (1+\lambda_i) = \theta \sum \pi_i \lambda_i.$$

Combine with equation (30) above, we can simplify to get

$$\left(1+\bar{\lambda}\right)(\nu_B - \nu) = \bar{\lambda}\theta\omega,$$

which is (29).

Lemma 3 *(Limited Commitment Constraint)*

Suppose $\omega = 1$, then the following conditions are equivalent:

1. *The limited commitment constraint (25) binds for $i = I$.*
2. *The limited commitment constraint (25) binds for $i = N$.*
3. *The following equation holds:*

$$(\nu_B - \theta) n_0 = (\theta + \nu - \nu_B) B_f \tag{31}$$

Proof. *Assume $\omega = 1$, then (28) implies $\frac{\nu_K}{Q^i} = \nu_B, \ \ \forall i$, that is, $Q^I = Q^N = 1$, where $Q^I = 1$ has to equal to one due to the absence of adjustment cost. Also, using (29), we have $\nu_K = \nu_B = 1 + \frac{\bar{\lambda}}{1+\bar{\lambda}}\theta\omega$. Constraint (25) can then be written as:*

$$(\nu_K - \theta)\left(K_1^i - B_B^i\right) \geq \nu B_f. \tag{32}$$

Note that with $Q^I = Q^N = 1$, the budget constraint implies $K_1^i - B_B^i = n_0 + B_f$, that is (25) does not depend on i. This proves the equivalence between 1) and 2).

To see the equivalence between 1), 2) and 3), note that by the above lemma, the *two constraints in (25) can be replaced by (32), which is equivalent to:*

$$(\nu_K - \theta)[n_0 + B_f] \geq \nu B_f,$$

which is (31). ■

With the above two lemmas, the equilibrium can be easily solved by the following three equations:

$$\begin{aligned} (\nu_B - \theta) n_0 &= (\theta + 1 - \nu_B) B_f \\ (1 - \delta) K_0 + I &= n_0 + B_f \\ \nu_B &= E[MA_1] = \frac{A_0 K_0 - I}{(1 - \delta) K_0 + I} \end{aligned}$$

to solve for three unknowns, (ν_B, B_f, I), which we summarize below:

Proposition 1 *(Equilibrium)*

Assume $\omega = 1$. Suppose $n_0 \geq n_0^*$, then first best price and allocations also constitute an equilibrium in the GK economy.

Suppose $n_0 < n_0^*$, then the equilibrium is given by:

$$\begin{aligned} \hat{I} &= \frac{1}{2 + \theta}[n_0 + A_0 - (1 + \theta)(1 - \delta) K_0] < I^* \\ B_f &= \frac{A_0 + (1 - \delta)}{2 + \theta} K_0 - \frac{1 + \theta}{2 + \theta} n_0 > B_f^* \\ B &= (1 - \pi) \hat{I} < B^* \\ \nu_B &= \nu_K = \frac{\left[A_0 K_0 - \hat{I}\right]}{\left[(1 - \delta) K_0 + \hat{I}\right]} > 1 \\ Q^I &= Q^N = 1 \end{aligned}$$

IV Adding capital misallocation

Setup of the model

- The household problem is the same as before. This gives the SDF:

$$M = \beta \left(\frac{C_0}{C_1} \right).$$

- Firm problem (final goods producer):

$$\max_{\{Y_j\}} \left\{ Y - \int_{[0,1]} p_j Y_j dj \right\} \Big|_{Y = \left[\int_{[0,1]} Y_j^{\frac{\eta-1}{\eta}} dj \right]^{\frac{\eta}{\eta-1}}}. \tag{33}$$

- Firm problem (intermediate good producer):

$$\begin{aligned} &\max \left\{ p_j Y_j - MPK_j \cdot K_j - MPL_t \cdot L_j \right\}, \\ &subject\ to: \qquad Y_j = \bar{A} z_j^{\frac{1}{\eta-1}} K_j^{\alpha} L_j^{1-\alpha}, \end{aligned} \tag{34}$$

where $\bar{A}_t$ is the aggregate productivity common across all islands and z_j is an island-j-specific idiosyncratic productivity shock that can take on two values, $\ln z_j \in \{\varepsilon_H, \varepsilon_L\}$ with $E[e^{\varepsilon}] = 1$. Recall that $\hat{A} = \left[\int A_j^{\eta-1} dj \right]^{\frac{1}{\eta-1}}$. Under our normalization, therefore, the first best agg productivity is $\bar{A}$.

- We also introduce variable capital utilization by specifying a capital storage technology in our model. We assume that the current period capital, K_t, can be used for two purposes: production of intermediate goods and storage. In the competitive market, the capital storage firm acquires K_S at the market price Q and saves the capital for next period through a storage technology $g(\cdot)$. The capital storage firm chooses K_S to maximize profit:

$$D_K = \max_{K_S} \left\{ g\left(\frac{K_S}{K} \right) K - Q K_S \right\}.$$

We use u to denote the capital utilization rate, that is, $u = 1 - \frac{K_S}{K}$. We assume that utilized capital depreciates linearly at rate δ. Therefore, the law of motion of capital is

$$K_1 = \left[g(1-u) + (1-\delta) u \right] K_0 + I_0. \tag{35}$$

We further assume that $g(0) = 0$, $g' > 1-\delta$, and $g'' < 0$ (which implies a concave storage technology). These assumptions together imply that depreciation is increasing in utilization, and unutilized capital depreciates at a lower rate than utilized capital. Our model is therefore a special case of the variable capital utilization model of Greenwood et al.

Note that the first-order condition for the capital-goods-producing firm implies

$$Q = g'(1-u). \tag{36}$$

Because the storage technology $g(\cdot)$ is concave, the market price of capital, Q, increases with the capital utilization rate, u.

- Bank's problem. Banks start with initial net worth N_0, they borrow B_0 from the households to purchase K_1 subject to the following budget constraint:

  $$K_1 = N_0 + B_0,$$

  where I used the fact that the price of capital equals one due to the absence of any adjustment cost. The net worth of the bank in the next period is given by:

  $$N_H = Q_H [K_1 + RA_H] - QRA_H - B_0 R_f,$$

  if $\varepsilon = \varepsilon_H$, and

  $$N_L = Q_L [K_1 + RA_L] - QRA_L - B_0 R_f,$$

  if $\varepsilon = \varepsilon_L$. Here Q_H is the price of capital on high productivity islands, Q_L is the price of capital on low productivity islands, and Q is the price of capital on the reallocation market. If $\varepsilon = \varepsilon_H$, the bank purchase RA_H amount of capital on the reallocation market at price Q. It sells to local firms at price Q_H

$$V(N_0) = \max_{B_0,K,RA_H,RA_L} E\left[M\left\{\begin{array}{c}\pi\left[Q_H\left(K+RA_H\right)-QRA_H-R_fB_0\right]\\ +\ (1-\pi)\left[Q_L\left(K+RA_L\right)-QRA_L-RB_0\right]\end{array}\right\}\right], \tag{37}$$

$$subject\ to: K = N_0 + B_0, \tag{38}$$

$$Q_H\left(K+RA_H\right)-QRA_H-R_fB_0 \geq \theta Q_H\left(K+RA_H\right), \tag{39}$$

$$Q_L\left(K+RA_L\right)-QRA_L-R_fB_0 \geq \theta Q_L\left(K+RA_L\right). \tag{40}$$

- Market clearing. First, the amount of capital used for production on all islands must sum to K_t, which is the total amount of utilized capital in the economy:

$$\pi\left(K+RA_H\right)+(1-\pi)\left(K+RA_L\right) = uK. \tag{41}$$

  Second, we need a market clearing condition for capital on each island. This market clearing condition implies that in equilibrium, the price of capital on each island must satisfy

$$Q_j = MPK_j + 1 - \delta,$$

  where MPK_j is the marginal product of capital on island j in period $t+1$. Third, the total net worth of the banking sector equals the sum of banks' net worth across all islands:

$$N_t = \int N_{j,t}dj. \tag{42}$$

  Fourth, labor market clearing requires $\int L_{j,t}dj = 1$ because we assume inelastic labor supply and normalize total labor endowment to one. Finally, market clearing for final goods requires that total consumption and investment sum to total output: $C+I=Y$, where Y is total output of final goods defined in (33).

Aggregation on the product market Using the results from the last lecture, we have shown that aggregate output can be written as $Y = \hat{A} \times EF \times (uK)^{\alpha} L^{1-\alpha}$, where $\hat{A} = \left[\int A_j^{\eta-1} dj\right]^{\frac{1}{\eta-1}}$ is the "first best" aggregate productivity, and

$$Y = \left\{ \int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}} L^{1-\alpha}; \;\; EF = \left\{ \frac{\int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj}{\left[\int A_j^{\eta-1} dj\right]^{1-\xi} K^{\xi}} \right\}^{\frac{1}{(\eta-1)(1-\xi)}}, \quad (43)$$

where $\xi = \frac{\alpha\left(1-\frac{1}{\eta}\right)}{1-(1-\alpha)\left(1-\frac{1}{\eta}\right)} = \frac{\alpha\eta-\alpha}{1+\alpha\eta-\alpha}$ (Note that this definition implies that $1-\xi = \frac{1}{1+\alpha\eta-\alpha}$ and $\frac{\xi}{1-\xi} = \alpha(\eta-1)$).

In our setup, $A_j = \bar{A} z_j^{\frac{1}{\eta-1}}$. Using the normalization condition $\int L_j dj = 1$, total output can be computed as

$$Y = \bar{A} \left\{ \int z_j^{(1-\xi)} (K_j + RA_j)^{\xi} dj \right\}^{\frac{\alpha}{\xi}} = \left\{ \int z_j^{(1-\xi)} \left(\frac{K_j + RA_j}{K} \right)^{\xi} dj \right\}^{\frac{\alpha}{\xi}} \bar{A} K^{\alpha}. \quad (44)$$

Under our assumption that there are only two possible realizations of idiosyncratic productivity shocks, we can further simplify the aggregate output function. Let RA_H and RA_L denote the capital reallocation on high- and low-productivity islands, respectively. Let $\phi = \frac{K+RA_H}{K+RA_L}$ be the ratio of firm size across islands. The resource constraint (41) can be written as $uK = \pi(K + RA_H) + (1-\pi)(K + RA_L)$, which, together with the definition of ϕ, implies

$$\frac{K + RA_H}{uK} = \frac{\phi}{\pi\phi + (1-\pi)}; \quad \frac{K + RA_L}{uK} = \frac{1}{\pi\phi + (1-\pi)}. \quad (45)$$

Together with Equation (44), we can write aggregate output as a function of (ϕ, u).

Also, our previous results imply that

$$MPK_j = \alpha \left\{ \int A_j^{(\eta-1)(1-\xi)} k_j^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}-1} L^{1-\alpha} \times A_j^{(\eta-1)(1-\xi)} k_j^{\xi-1}.$$

In our notation,

$$\begin{aligned} MPK_j &= \alpha \bar{A} z_j^{(1-\xi)} K_j^{\xi-1} \left\{ \int z_j^{(1-\xi)} K_j^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}-1} \\ &= \alpha \bar{A} z_j^{(1-\xi)} \left(\frac{K_j}{uK} \right)^{\xi-1} (uK)^{\xi-1} \left\{ \int z_j^{(1-\xi)} \left(\frac{K_j}{uK} \right)^{\xi} (uK)^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}-1} \\ &= \alpha \bar{A} (uK)^{\alpha-1} z_j^{(1-\xi)} \left(\frac{K_j}{uK} \right)^{\xi-1} \left\{ \int z_j^{(1-\xi)} \left(\frac{K_j}{uK} \right)^{\xi} dj \right\}^{\frac{1}{(\eta-1)(1-\xi)}-1} . \end{aligned}$$

If we assume $\bar{A} = AK^{1-\alpha}$, where A is an exogenous productivity, then the above expressions simplify and we can summarize our results as follows.

Proposition 2 *(Aggregation of the Product Market)*

The total output of the economy at period 1 is $Y = Au^{\alpha} f(\phi) K$, *where the function* $f : \left[1, \hat{\phi}\right] \to [0,1]$ *is defined as*

$$f(\phi) = \left\{ \pi e^{(1-\xi)\varepsilon_H} \left(\frac{\phi}{\pi\phi + (1-\pi)} \right)^{\xi} + (1-\pi) e^{(1-\xi)\varepsilon_L} \left(\frac{1}{\pi\phi + (1-\pi)} \right)^{\xi} \right\}^{\frac{\alpha}{\xi}} \quad (46)$$

and $\hat{\phi} = e^{\varepsilon_H - \varepsilon_L}$.

The marginal product of capital on low- and high-productivity islands, MPK_L *and* MPK_H, *can be written as functions of* (A, ϕ, u):

$$MPK_L(A, \phi, u) = \alpha A u^{1-\alpha} f(\phi) \frac{\pi\phi + 1 - \pi}{\pi e^{(1-\xi)(\varepsilon_H - \varepsilon_L)} \phi^{\xi} + 1 - \pi}, \quad (47)$$

$$MPK_H(A, \phi, u) = MPK_L(A, \phi, u) \frac{e^{(1-\xi)(\varepsilon_H - \varepsilon_L)}}{\phi^{1-\xi}}. \quad (48)$$

It is not hard to show that, given our normalization condition $E\left[e^{\varepsilon_{j,t+1}}\right] = 1$, the efficient level of ϕ that implies equalization of the marginal production of capital across all islands is $\hat{\phi} = e^{\varepsilon_H - \varepsilon_L}$ and $f\left(\hat{\phi}\right) = 1$. The function $f(\phi)$ is a measure of the efficiency of capital reallocation, since ϕ can be interpreted as a measure of capital reallocation. It is straightforward to show that f is increasing in ϕ.

Financial constraints Dividing both sides of constraints (39) and (40) by K and using equation (45), we obtain

$$(1-\theta) Q_H(A,\phi) + [(1-\theta) Q_H(A,\phi) - Q(u)] \left[\frac{\phi u}{\pi\phi + (1-\pi)} - 1\right] \geq s, \quad (49)$$

$$(1-\theta) Q_L(A,\phi) + [(1-\theta) Q_L(A,\phi) - Q(u)] \left[\frac{u}{\pi\phi + (1-\pi)} - 1\right] \geq s, \quad (50)$$

where we denote $s = \frac{R_f B_0}{K}$ as the ratio of bank liability to capital.

Let ζ_H and ζ_L denote the Lagrangian multipliers on the limited enforcement constraints, (39) and (40). The first-order conditions with respect to RA_j can be used to derive a relationship between Lagrangian multipliers and the prices of capital on the high- and low-productivity islands. We use this relationship to define

$$\zeta_H(A,\phi,u) = \frac{\pi [Q_H(A,\phi,u) - Q(u)]}{Q(u) - (1-\theta) Q_H(A,\phi,u)} \geq 0, \quad > 0 \Longrightarrow (49) \textit{ holds with } \text{“} = \text{”}, \quad (51)$$

$$\zeta_L(A,\phi,u) = \frac{(1-\pi) [Q_L(A,\phi,u) - Q(u)]}{Q(u) - (1-\theta) Q_L(A,\phi,u)} \geq 0 > 0 \Longrightarrow (50) \textit{ holds with } \text{“} = \text{”}, \quad (52)$$

which imply that

$$Q_H - Q \geq 0, \quad > 0 \Longrightarrow (49) \textit{ holds with } \text{“} = \text{”}, \quad (53)$$

$$Q_L - Q \geq 0, \quad > 0 \Longrightarrow (50) \textit{ holds with } \text{“} = \text{”}. \quad (54)$$

If both of the limited enforcement constraints (49) and (50) hold with equality, then they jointly determine ϕ and u as functions of (A,s). If neither (49) nor (50) is binding, then $\zeta_H(A,\phi,u) = \zeta_L(A,\phi,u) = 0$, implying $Q_H(A,\phi,u) = Q_L(A,\phi,u) = Q(u)$. Again, ϕ and u can be determined as functions of (A,s). In general, Equations (49), (50), (51), (52), and the complementary slackness condition determine ϕ and u as functions of (A,s), which we will denote as $\phi(A,s)$ and $u(A,s)$. The following proposition builds on this observation and characterizes the nature of the binding constraints.

Proposition 3 *(Characterization of Binding Constraints) There exist* $\hat{s}(A)$ *and* $\bar{s}(A)$

such that

1. *If $s \leq \hat{s}(A)$, then none of the limited commitment constraints bind, and $\phi(A, s)$ and $u(A, s)$ are determined by the equality versions of (53) and (54).*

2. *If $\hat{s}(A) < s \leq \bar{s}(A)$, then the limited commitment constraint for banks on high productivity islands binds, and $\phi(A, s)$ and $u(A, s)$ are determined by the equality versions of (49) and (54).*

3. *If $s > \bar{s}(A)$, then the limited commitment constraint for all banks binds, and $\phi(A, s)$ and $u(A, s)$ are determined by the equality versions of (49) and (50).*

4. *Both $\bar{s}(A)$ and $\hat{s}(A)$ decrease with A.*

Model Solution To solve the model, we first note that no arbitrage implies that banks must be indifferent between investing in one unit of capital to sell on the reallocation market and saving at the risk-free interest rate. Therefore, $R_f = Q(u)$. Second, intertemporal optimality on the consumer side implies that $R_f = \frac{C_1}{\beta C_0}$. Using the aggregate resource constraints to replace C_0 and C_1 and write ϕ and u as functions of (A, s), we obtain

$$\frac{Au(A,s)^{\alpha} f(\phi(A,s))\left[(1-\delta)K_0 + I_0\right]}{A_0 K_0 - I_0} = Q(u(A,s)). \tag{55}$$

Third, using the definition of s, we can replace B_0 in banks' budget constraint (38) by $B_0 = \frac{s}{R_f}\left[(1-\delta)K_0 + I_0\right]$ and obtain

$$N_0 = \left[(1-\delta)K_0 + I_0\right]\left(1 - \frac{s}{R_f}\right). \tag{56}$$

Note that given the initial condition (A_0, A, K_0, N_0), Equations (55) and (56) jointly determine investment I_0 and the bank liability-to-capital ratio, s.