Lecture 3: Recursive Policy Iteration

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Abstract

I describe some simple variations of the GK model. First, I want to clarify some subtle timing issue in the model. Second, I want to use these simple setups to understand some of the basic tradeoffs of the model.

I A Warm-up example: the RBC model

To understand the idea of recursive policy iteration, let start with the familar RBC model:

$$\begin{aligned} V(A,K) &= \max u(C) + \beta E\left[V(A',K')\right] \\ K' &= (1-\delta)K + I, \\ C + H(I,K) &= Af(K), \end{aligned} \tag{1}$$

where $f(K)$ is the production function and $H(I,K)$ is a concave and CRS adjustment cost function. To obtain the global solution, the standard way to sovle the model is value function iteration, which relies on the standard Bellman operator being a contraction.

Alternatively, the solution of the problem can be characterized by the policy functions $\{C(A,K), I(A,K)\}$ and the partial derivative of the value function $V_K(A,K)$ that satisfy the optimality conditions (the first order condition, the resource constraint, and the envelop condition):

$$u_C(C(A,K))\,H_I(I(A,K),K) = \beta E\left[V_K(A',K')\right] \tag{2}$$

$$V_K(A,K) = u_C(C(A,K))\left[Af_K(K) + H_K(I(A,K),K)\right] + \beta(1-\delta)E\left[V_K(A',K')\right] \tag{3}$$

$$C(A,K) + I(A,K) = Af(K) \tag{4}$$

Equations (2)-(4) are three functional equations that determine three functions, $C(A,K)$, $I(A,K)$, and $V_K(A,K)$. We can combine (2) and (3) to write:

$$\begin{aligned} & u_C(C(A,K))\,H_I(I(A,K),K) \\ = \; & \beta E\left\{u_C(C(A',K'))\left[A'f_K(K') + H_K(I(A',K'),K') + (1-\delta)H_I\;(I(A',K'),K)\right]\right\} \end{aligned} \tag{5}$$

The idea of policy function iteration is to work with (5), instead of the programming problem (1). Here is a standard procedure. Start with an initial guess of the consumption policy, $C^n(A,K)$. Given the guess of the consumption policy function in the nth iteration, let $I^n(A,K)$ be the corresponding investment policy that solves $C^n(A,K)+I^n(A,K)=Af(K)$. We can rewrite (5) as:

$$\begin{aligned} & u_C(C(A,K))\,H_I(I(A,K),K) \\ = & \beta E\left\{u_C(C^n(A',K'))\left[A' f_K(K')+H_K(I^n(A',K'),K')+(1-\delta)H_I\ (I^n(A',K'),K)\right]\right\} \end{aligned} \tag{6}$$

where $K'=(1-\delta)K+I$. We solve the above functional equation for the policy functions $C(A,K)$ and $I(A,K)$ that satisfies (6). Update the policy function: $C^{n+1}(A,K)=C(A,K)$, $I^{n+1}(A,K)=I(A,K)$ and iterate this procedure until convergence. Note here, we have tranformed an optimization problem to a problem of solve a nonlinear (functional) equation. We can combine this with the endogenous grid method to speed up.

The following equivalent method has an obvious asset pricing interpretation. We define $q(A,K)=H_I(I(A,K),K)$ and rewrite equation (5) as:

$$q(A,K)=E\left[\frac{\beta u'(C(A',K'))}{u'(C(A,K))}\left\{\left[A' f_K(K')+H_K(I(A',K'),K')\right]+\beta(1-\delta)q(A',K')\right\}\right], \tag{7}$$

which is a present value relationship. We can start with an initial guess of $q^n(A',K')$, use $q(A,K)=H_I(I,K)$ and $C+H(I,K)=Af(K)$ to solve for the corresponding $C^n(A,K)$ and $I^n(A,K)$. Solve for $\{C(A,K),I(A,K),q(A,K)\}$ that satisfies

$$q(A,K)=E\left[\frac{\beta u'(C^n(A',K'))}{u'(C(A,K))}\left\{\left[A' f_K(K')+H_K(I^n(A',K'),K')\right]+\beta(1-\delta)q^n(A',K')\right\}\right]$$

along with the equations $q(A,K)=H_I(I,K)$ and $C+H(I,K)=Af(K)$. It is reasonable to expect this procedure to be a contraction (although a proof may be hard), because there are build-in discountings in this procedure.

II Financial frictions and capital misallocation

A Setup of the model

Household problem Household's utility maximization problem can be written in a recursive fashion:

$$V(z, W) = \max_{\{C, B_f\}} (1-\beta) \ln C + \beta E\left[V(z', W') | z\right]$$

subject to:

$$C + B_f = W + MPL(z) \cdot l$$

$$W' = B_f R_f(z) + \int D_B(j)(z')\, dj + D_K(z')$$

As before, markets are incomplete. In particular, HH can only save through a risk-free asset.

The HH problem gives the intertemporal Euler equation:

$$E\left[M(z, z') | z\right] R_f(z) = 1$$

in which

$$M(z, z') = \beta \left[\frac{C(z')}{C(z)}\right]^{-1}$$

Output producers There are three types of non-financial firms in our model, namely, intermediate goods producers, final goods producers and capital goods producers.

- The profit maximization problem of the final goods producer:

$$\max_{y_j} \left\{ Y - \int_{[0,1]} p_j y_j dj \right\} \Bigg|_{Y = \left[\int_{[0,1]} y_j^{\frac{\eta-1}{\eta}} dj\right]^{\frac{\eta}{\eta-1}}} .$$

- The profit maximization problem for the intermediate goods producer on island

j:

$$\max \left\{ p_j Y_j - MPK_j \cdot K_j - MPL \cdot L_j \right\},$$
$$subject\ to: \qquad Y_j = \bar{A}_t z_j^{\frac{1}{\eta-1}} K_j^{a} L_j^{1-\alpha}, \tag{8}$$

where $\bar{A}$ is the aggregate productivity common across all islands and z_j is an island-j-specific idiosyncratic productivity shock that follows:

$$\ln z_{j,t+1} = \ln z_{j,t} + \varepsilon_{j,t+1}. \tag{9}$$

In the above equation, $\varepsilon_{j,t+1}$ is i.i.d. across firms and over time with $E\left[e^{\varepsilon_{j,t+1}}\right] = 1$. The assumption that $E\left[e^{\varepsilon_{j,t+1}}\right] = 1$ is a normalization, which implies that the average of idiosyncratic productivity of all firms is constant over time, so that aggregate productivity growth comes entirely from $\bar{A}_t$ and not from the growth of idiosyncratic shocks $\varepsilon_{j,t}$. We assume that $\varepsilon_{j,t+1}$ can take on two values, ε_H and ε_L. Recall that $\hat{A} = \left[\int A_j^{\eta-1} dj\right]^{\frac{1}{\eta-1}}$. Under our normalization, the first best agg productivity is $\bar{A}_t$.

- We also introduce variable capital utilization by specifying a capital storage technology in our model. We assume that the current period capital, K, can be used for two purposes: production of intermediate goods and storage. In the competitive market, the capital storage firm acquires K_S at the market price Q_t and saves the capital for next period through a storage technology $g(\cdot)$. The capital storage firm chooses K_S to maximize profit:

$$D_K = \max_{K_S} \left\{ g\left(\frac{K_S}{K}\right) K - QK_S \right\}.$$

We use u to denote the capital utilization rate, that is, $u = 1 - \frac{K_S}{K}$. We assume that utilized capital depreciates linearly at rate δ. Therefore, the law of motion of capital is

$$K' = \left[g\left(1-u\right) + \left(1-\delta\right)u\right]K + I. \tag{10}$$

We further assume that $g\left(0\right) = 0$, $g' > 1-\delta$, and $g'' < 0$ (which implies a concave storage technology). These assumptions together imply that depreciation is increasing in utilization, and unutilized capital depreciates at a lower rate than utilized capital. Our model is therefore a special case of the variable capital utilization model of Greenwood et al.

Below we summarize the key equilibrium conditions on the production side of the economy. First, the optimality condition for capital goods producers implies

$$Q = g'(1-u). \tag{11}$$

Second, we will make assumptions so that the marginal product of capital on islands that realizes the same ε_{j+1} shocks are the same. That is, the tightness of financial constraints does not depend on the history of idiosyncratic productivity shocks. In this case, aggregate output and the marginal product of capital can be computed as follows.

Proposition 1 *(Aggregation of the Product Market)*

The total output of the economy at period 1 is $Y = Au^{\alpha} f(\phi) K$*, where the function* $f : \left[1, \hat{\phi}\right] \rightarrow [0,1]$ *is defined as*

$$f(\phi) = \left\{ \pi e^{(1-\xi)\varepsilon_H} \left(\frac{\phi}{\pi\phi + (1-\pi)} \right)^{\xi} + (1-\pi) e^{(1-\xi)\varepsilon_L} \left(\frac{1}{\pi\phi + (1-\pi)} \right)^{\xi} \right\}^{\frac{\alpha}{\xi}} \tag{12}$$

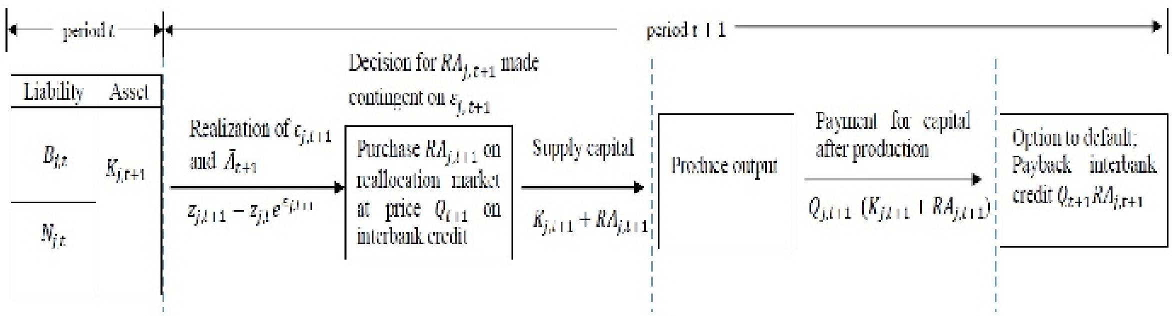
and $\hat{\phi} = e^{\varepsilon_H - \varepsilon_L}$*.*

The marginal product of capital on low- and high-productivity islands, MPK_L *and* MPK_H*, can be written as functions of* (A, ϕ, u)*:*

$$MPK_L(A, \phi, u) = \alpha A u^{1-\alpha} f(\phi) \frac{\pi\phi + 1 - \pi}{\pi e^{(1-\xi)(\varepsilon_H - \varepsilon_L)} \phi^{\xi} + 1 - \pi}, \tag{13}$$

$$MPK_H(A, \phi, u) = MPK_L(A, \phi, u) \frac{e^{(1-\xi)(\varepsilon_H - \varepsilon_L)}}{\phi^{1-\xi}}. \tag{14}$$

It is not hard to show that, given our normalization condition $E\left[e^{\varepsilon_{j,t+1}}\right] = 1$, the efficient level of ϕ that implies equalization of the marginal production of capital



across all islands is $\hat{\phi} = e^{\varepsilon_H - \varepsilon_L}$ and $f\left(\hat{\phi}\right) = 1$. The function $f(\phi)$ is a measure of the efficiency of capital reallocation, since ϕ can be interpreted as a measure of capital reallocation. It is straightforward to show that f is increasing in ϕ.

Bank's problem There is a continuum of islands with i.i.d. productivity shocks, and there is one competitive financial intermediary on each island. Consider a bank on island j who enters into period t with initial net worth $N_{j,t}$. It chooses the total amount of borrowing from the household, $B_{j,t}$, and the total amount of capital stock for the next period, $K_{j,t+1}$ subject to the following budget constraint:

$$K_{j,t+1} = N_{j,t} + B_{j,t}. \tag{15}$$

The total amount of capital for the next period, $K_{j,t+1}$, is determined at the end of period t, before the realization of shocks in the next period (one-period time to plan). However, different from the standard representative firm setup, capital can be reallocated across firms after shocks are realized. The market for capital reallocation opens after the realization of the aggregate productivity shock $\bar{A}_{t+1}$ and the island-j-specific idiosyncratic productivity shock, $\varepsilon_{j,t+1}$. Figure A illustrates the timing of events period t and in period $t+1$.

If the bank pays back its interbank loans and household deposits, the total net worth is

$$N_{j,t+1} = Q_{j,t+1}\left[K_{j,t+1} + RA_{j,t+1}\right] - Q_{t+1}RA_{j,t+1} - R_{f,t+1}B_{j,t}. \tag{16}$$

The possibility of default implies that the contracting between borrowing and lending banks must respect the following limited enforcement constraint:

$$N_{j,t+1} \geq \theta Q_{j,t+1}\left[K_{j,t+1} + RA_{j,t+1}\right], \forall\ t\ and\ \forall\ j. \tag{17}$$

The incentive compatibility constraint implies that in anticipating the possibility of default, lending banks ensure that the borrowing banks do not have an incentive to default on loans in all possible states of the world. Combining Equations (16) and (17), we can write the limited enforcement constraint as

$$(1-\theta)\,Q_{j,t+1}\left[K_{j,t+1} + RA_{j,t+1}\right] - Q_{t+1}RA_{j,t+1} \geq R_{f,t+1}B_{j,t}. \tag{18}$$

Note that the left-hand side of (18) depends on the realization of idiosyncratic shocks, and the constraint has to hold for all possible realizations of idiosyncratic and aggregate shocks. Because more productive firms typically need to borrow from less productive ones, the financial constraint limits not only the total amount of borrowing of the corporate sector from the household sector, but also, and more importantly, the amount of borrowing within the corporate sector across firms.

Note that $Q_{j,t}$ is the price of capital on island j. The market clearing condition for capital on each island implies

$$Q_{j,t} = MPK_{j,t} + 1 - \delta,$$

where $MPK_{j,t}$ is the marginal product of capital on island j in period t.

Market clearing First, the amount of capital used for production on all islands must sum to u_tK_t, which is the total amount of utilized capital in the economy:

$$\int \left(K_{j,t} + RA_{j,t}\right) dj = u_tK_t. \tag{19}$$

Under the assumption that there are only two possible realizations of idiosyncratic productivity shocks, the above equation can be written as $uK = \pi\left(K + RA_H\right) + (1-\pi)\left(K + RA_L\right)$. Let RA_H and RA_L denote the capital reallocation on high- and low-productivity islands, respectively. Let $\phi = \frac{K+RA_H}{K+RA_L}$ be the ratio of firm size across islands. The resource constraint implies:

$$\frac{K + RA_H}{uK} = \frac{\phi}{\pi\phi + (1-\pi)}; \quad \frac{K + RA_L}{uK} = \frac{1}{\pi\phi + (1-\pi)}. \tag{20}$$

Second, the total net worth of the banking sector equals the sum of banks' net worth across all islands:

$$N_t = \int N_{j,t} dj. \tag{21}$$

Third, labor market clearing requires $\int L_{j,t} dj = 1$ because we assume inelastic labor supply and normalize total labor endowment to one.

Finally, market clearing for final goods requires that total consumption and investment sum to total output: $C_t + I_t = Y_t$.

III Solution

Banker optimization Other parts of the model are standard, except for the banks' optimization problem:

$$V(z_t, N_{j,t}) = \max_{\{K_{j,t+1}, B_{j,t}, RA_{j,t+1}(\varepsilon_{j,t+1})\}} E_t \left[M_{t+1} \left\{ (1 - \Lambda_{t+1}) N_{j,t+1} + \Lambda_{t+1} V_{t+1}(z_{t+1}, N_{j,t+1}) \right\} \right]$$

$$q_t K_{j,t+1} = N_{j,t} + B_{j,t}$$

$$N_{j,t+1} = Q_{j,t+1} \left[K_{j,t+1} + RA_{j,t+1} \right] - Q_{t+1} RA_{j,t+1} - R_{f,t+1} B_{j,t},$$

$$(1-\theta) Q_{H,t+1} \left[K_{j,t+1} + RA_{j,t+1}(H) \right] - Q_{t+1} RA_{j,t+1}(H) \geq R_{f,t+1} B_{j,t}, \tag{22}$$

$$(1-\theta) Q_{H,t+1} \left[K_{j,t+1} + RA_{j,t+1}(H) \right] - Q_{t+1} RA_{j,t+1}(H) \geq R_{f,t+1} B_{j,t}. \tag{23}$$

We use $\xi(z')$ and $\xi(z')$ to denote the lagrangian multipliers of the limited commitment constraints. The lagrangian for bank's problem is

$$\mu(z) N = E \left[\widetilde{M'} \left\{ \begin{array}{c} \pi \left[Q_H(z') \frac{N+B_f}{q(z)} + \left[Q_H(z') - Q(z') \right] RA_H(z') - R_f(z) B_f \right] \\ + (1-\pi) \left[Q_L(z') \frac{N+B_f}{q(z)} + \left[Q_L(z') - Q(z') \right] RA_L(z') - R_f(z) B_f \right] \\ + \pi \zeta_H(z') \left[(1-\theta) Q_H(z') \frac{N+B_f}{q(z)} - O_H(z') RA_H(z') - R_f(z) B_f \right] \\ + (1-\pi) \zeta_L(z') \left[(1-\theta) Q_L(z') \frac{N+B_f}{q(z)} - O_L(z') RA_L(z') - R_f(z) B_f \right] \end{array} \right\} \right]$$

where we used the following notation to simply the expressions:

$$O_H(z') \overset{def}{\equiv} Q(z') - (1-\theta) Q_H(z'),$$

$$O_L(z') \overset{def}{\equiv} Q(z') - (1-\theta) Q_L(z').$$

The first order condition with respect to $\{RA_H(z'), RA_L(z')\}_{z'}$ implies (Note that $\{RA_H(z')\}_{z'}$ is a vector of choice variables, one for each z'):

$$\zeta_H(z') = \frac{Q_H(z') - Q(z')}{Q(z') - (1-\theta) Q_H(z')} \geq 0, \quad > 0 \implies (22) \ \textit{holds with} \ =, \tag{24}$$

$$\zeta_L(z') = \frac{Q_L(z') - Q(z')}{Q(z') - (1-\theta) Q_L(z')} \geq 0, \quad > 0 \implies (23) \ \textit{holds with} \ =, \tag{25}$$

Suppose $Q_H(z') > Q_L(z')$, what can you conclude about $\zeta_H(z')$ and $\zeta_L(z')$? What is the intuition? Note that (24) and (25) can be written as:

$$\begin{aligned} Q_H(z') - Q(z') &= \zeta_H(z') \{Q(z') - (1-\theta) Q_H(z')\}, \\ Q_L(z') - Q(z') &= \zeta_L(z') \{Q(z') - (1-\theta) Q_L(z')\}, \end{aligned}$$

or equivalently,

$$\begin{aligned} [1 + (1-\theta) \zeta_H(z')] Q_H(z') &= [1 + \zeta_H(z')] Q(z'); \\ [1 + (1-\theta) \zeta_L(z')] Q_L(z') &= [1 + \zeta_L(z')] Q(z') \end{aligned}$$

together they imply:

$$\begin{aligned} & \pi [1 + (1-\theta) \zeta_H(z')] Q_H(z') + (1-\pi) [1 + (1-\theta) \zeta_L(z')] Q_L(z') \\ = & \ \{1 + \pi \zeta_H(z') + (1-\pi) \zeta_L(z')\} Q(z') \end{aligned} \tag{26}$$

That is, banks must be indifferent between selling the capital on their own islands and selling their capital to other islands (on the reallocation market).

The FOC with respect to B_f:

$$\begin{aligned} & q(z) E\left[\widetilde{M}'\left\{1+\pi\zeta_H(z')+(1-\pi)\zeta_L(z')\right\}\right] R_f(z) \\ = \; & E\left[\widetilde{M}'\left\{\pi\left[1+(1-\theta)\zeta_H(z')\right]Q_H(z')+(1-\pi)\left[1+(1-\theta)\zeta_L(z')\right]Q_L(z')\right\}\right], \end{aligned}$$

Note that using (26), the above imply:

$$E\left[\widetilde{M}'\left\{1+\pi\zeta_H(z')+(1-\pi)\zeta_L(z')\right\}\right] R_f(z) = E\left[\widetilde{M}'\left\{1+\pi\zeta_H(z')+(1-\pi)\zeta_L(z')\right\}\frac{Q(z')}{q(z)}\right], \tag{27}$$

Finally, the envelope condition implies

$$\begin{aligned} \mu(z) &= E\left[\widetilde{M}'\left\{\begin{array}{c} \pi\frac{Q_H(z')}{q(z)}+(1-\pi)\frac{Q_L(z')}{q(z)}+ \\ \pi\zeta_H(z')(1-\theta)\frac{Q_H(z')}{q(z)}+(1-\pi)\zeta_L(z')(1-\theta)\frac{Q_L(z')}{q(z)} \end{array}\right\}\right] \\ &= \frac{1}{q(z)}E\left[\widetilde{M}'\left\{\pi\left[1+(1-\theta)\zeta_H(z')\right]Q_H(z')+(1-\pi)\left[1+(1-\theta)\zeta_L(z')\right]Q_L(z')\right\}\right] \\ &= \frac{1}{q(z)}E\left[\widetilde{M}'\left\{1+\pi\zeta_H(z')+(1-\pi)\zeta_L(z')\right\}Q(z')\right], \end{aligned} \tag{28}$$

where the third equality again uses (26). We can simplify (27) and (28) to get:

$$\mu(z) = E\left[\widetilde{M}'\left\{1+\pi\zeta_H(z')+(1-\pi)\zeta_L(z')\right\}\right] R_f(z) \quad (saving\ through\ risk-free) \tag{29}$$

$$\mu(z) = \frac{1}{q(z)}E\left[\widetilde{M}'\left\{1+\pi\zeta_H(z')+(1-\pi)\zeta_L(z')\right\}Q(z')\right] \quad (saving\ through\ capital) \tag{30}$$

Capital reallocation and capital utilization Dividing both sides of constraints (22) and (23) by K and using equation (20), we obtain

$$(1-\theta)Q_H(A,\phi,u)+\left[(1-\theta)Q_H(A,\phi,u)-Q(u)\right]\left[\frac{\phi u}{\pi\phi+(1-\pi)}-1\right] \geq s, \tag{31}$$

$$(1-\theta)Q_L(A,\phi,u)+\left[(1-\theta)Q_L(A,\phi,u)-Q(u)\right]\left[\frac{u}{\pi\phi+(1-\pi)}-1\right] \geq s, \tag{32}$$

where we denote $s=\frac{R_f B}{K}$ as the ratio of bank liability to capital.

The Kuhn-Tucker conditions (24) and (25) implies

$$Q_H(A,\phi,u) - Q(u) \geq 0, \quad > 0 \Longrightarrow (31) \textit{ holds with } \text{“}=\text{”}, \tag{33}$$

$$Q_L(A,\phi,u) - Q(u) \geq 0 > 0 \Longrightarrow (32) \textit{ holds with } \text{“}=\text{”}. \tag{34}$$

Together, Equations (31), (32), (33), and (34) determine ϕ and u as functions of (A, s), which we will denote as $\phi(A,s)$ and $u(A,s)$. We summarize this in the following Proposition.

Proposition 2 *(Characterization of Binding Constraints) There exist $\hat{s}(A)$ and $\bar{s}(A)$ such that*

1. *If $s \leq \hat{s}(A)$, then none of the limited commitment constraints bind, and $\phi(A,s)$ and $u(A,s)$ are determined by the equality versions of (**??**) and (**??**).*
2. *If $\hat{s}(A) < s \leq \bar{s}(A)$, then the limited commitment constraint for banks on high productivity islands binds, and $\phi(A,s)$ and $u(A,s)$ are determined by the equality versions of (31) and (**??**).*
3. *If $s > \bar{s}(A)$, then the limited commitment constraint for all banks binds, and $\phi(A,s)$ and $u(A,s)$ are determined by the equality versions of (31) and (32).*
4. *Both $\bar{s}(A)$ and $\hat{s}(A)$ decrease with A.*

Summary of equilibrium conditions

1. The intertemporal Euler equation for households' intertemporal investment problem:
$$E\left[M(z,z')\right| z,s'] R_f(z;s') = 1, \tag{35}$$
where the stochastic discount factor is given by:
$$M(z,z') = \frac{\beta\left[Au^{\alpha}(z) f(\phi(z)) - i(z)\right]}{c(z')\left[g(1-u(z)) + (1-\delta)u(z) + i(z)\right]}. \tag{36}$$

2. Banks' optimal choice for intertemporal investment implies

$$\mu(z) = E\left[\widetilde{M}(z, z')\{1 + (\zeta_H(A', \phi(z'), u(z')) + \zeta_L(A', \phi(z'), u(z')))\} Q(u')\right], \tag{37}$$

where $\widetilde{M}(\omega, \omega')$ is defined as

$$\widetilde{M}(z, z') = M(z, z')\{1 - \Lambda' + \Lambda' \mu(z')\}. \tag{38}$$

3. The envelope condition on banks' optimization problem is

$$\mu(z) = E\left[\widetilde{M}(z, z')\{1 + \zeta_H(A', \phi(z'), u(z')) + \zeta_L(A', \phi(z'), u(z'))\}\right] R_f(z; s'). \tag{39}$$

4. Finally, we note that the resource constraint requires

$$c(z) + i(z) = A u^{\alpha}(z) f(\phi(z)). \tag{40}$$

We still need to define a state variable and figure out the law of motion of the state variable. This literature typically use net worth as the state variable, so let's start with *aggregate* net worth. Let N be the aggregate banking sector net worth in the current period. In the next period, a π fraction of all banks realizes $\varepsilon_{j,t+1} = \varepsilon_H$, and their total net worth becomes

$$\pi\left[Q_{H,t+1}\left[K_{t+1} + RA_{H,t+1}\right] - Q_{t+1} RA_{H,t+1} - R_{f,t+1} B_t\right].$$

A $1 - \pi$ fraction realizes a ε_L shock and their total net worth becomes

$$(1 - \pi)\left[Q_{L,t+1}\left[K_{t+1} + RA_{L,t+1}\right] - Q_{t+1} RA_{L,t+1} - R_{f,t+1} B_t\right].$$

Taking into account the fact that a fraction $1 - \Lambda_{t+1}$ of all banks are forced to liquidate, we then have

$$N_{t+1} = \Lambda_{t+1} \left\{ \begin{array}{c} \left[\pi Q_{H,t+1} \left(K_{t+1} + RA_{H,t+1}\right) + (1-\pi) Q_{L,t+1} \left(K_{t+1} + RA_{L,t+1}\right)\right] \\ -Q_{t+1} \left[\pi RA_{H,t+1} + (1-\pi) RA_{L,t+1}\right] - R_{f,t+1} B_t \end{array} \right\}. \tag{41}$$

We think of t as the current period and $t+1$ as the next period, and use the above relationship to express equilibrium quantities not as functions of time but rather as functions of state variables. We use lowercase letters to denote current period normalized quantities and lowercase letters with a prime ($'$) to denote next period normalized quantities. Using the fact that the capital income share is α, we obtain $\pi Q_{H,t+1} \left(K_{t+1} + RA_{H,t+1}\right) + (1-\pi) Q_{L,t+1} \left(K_{t+1} + RA_{L,t+1}\right) = \alpha Y_{t+1} + (1-\delta) K_{t+1}$. Using the resource constraint (19) and dividing both sides of Equation (41) by K_{t+1}, we obtain an expression for next period net worth normalized by the capital stock, n':

$$n' = \Lambda' \left\{\alpha A' (u')^{\alpha} f(\phi') + (1-u') MPK(u') + (1-\delta) - s\right\}, \tag{42}$$

where $MPK(u)$ is defined by

$$MPK(u) = Q(u) - (1-\delta), \tag{43}$$

with $Q(u)$ given in Equation (11).

We will show below that a much more convenient choice of the state variable is $s_{t+1} = \frac{R_{f,t+1} B_t}{K_{t+1}}$. Using normalized quantities,

$$s' = \frac{R_f b_f}{g(1-u) + (1-\delta) u + i}. \tag{44}$$

To obtain an expression for b_f, we can divide both sides of the bank budget constraint (15) by K:

$$g(1-u) + (1-\delta) u + i = n + b_f \tag{45}$$

and use equation (45) to replace b_f in (44) to obtain

$$s' = \frac{g(1-u) + (1-\delta) u + i - n}{g(1-u) + (1-\delta) u + i} R_f. \tag{46}$$

Using Equation (42) to replace n in (46), we obtain

$$s' = R_f(z; s') \left\{ 1 - \frac{\Lambda \left\{ \alpha \left(1 - \frac{1}{\eta}\right) Au(z) f(\phi(z)) + (1 - u(z)) MPK(u(z)) + (1 - \delta) - s \right\}}{g(1 - u(z)) + (1 - \delta) u(z) + i(z)} \right\}. \tag{47}$$

Recursive policy iteration First, we A typical step in the iterative procedure is to solve for $\{c(z), \mu(z)\}$ given an initial guess of the equilibrium functionals $\{c^n(z'), \mu^n(z')\}$. Given the initial guess, the SDF for log preference can be written as

$$M(z, z') = \beta \frac{C(z)}{C(z')} = \frac{\beta c(x, s)}{c^n(x', s')} \frac{K}{K'} = \frac{\beta [m(z) - i]}{c^n(x', s') [1 - \delta(z) + i(z)]},$$

where we denote

$$\begin{aligned} m(z) &= Au^{\alpha}(z) f(\phi(z)), \\ 1 - \delta(z) &= g(1 - u(z)) + (1 - \delta) u(z). \end{aligned}$$

Because the risk-free interest rate $R_f(z; s')$ satisfies Equation (35), we have:

$$R_f(z; s') = \frac{[1 - \delta(z) + i(z)]}{\beta [m(z) - i(z)] E\left[\frac{1}{c^n(z')} \middle| z, s' \right]}. \tag{48}$$

The law of motion of s in equation (47) can therefore be written as

$$s' = \frac{[1 - \delta(z) + i(z)] - \Lambda \{\alpha m(z) + (1 - u(z)) MPK(u(z)) + (1 - \delta) - s\}}{E\left[\frac{1}{c^n(z')} \middle| z, s' \right] \beta [m(z) - i(z)]}. \tag{49}$$

For each z, equation (49) is a nonlinear function of s' and $i(z)$. Below, we use equilibrium optimality conditions to derive another equation that can be used to jointly solve for the law of motion of s and the policy function $i(z)$.

Second, we will derive another equation that can be used to solve for $s', i(z)$. This equation, naturlly, comes from optimality conditions. Given the policy functions $\phi(z)$ and $u(z)$, we can represent the prices for capital and the Lagrangian multipliers as functions of the state variable z by using Equations (11), (13), (14), (31), (32), (33),

(34) and (43). With a slight abuse of notation, we denote these pricing functionals as $\{MPK_j(z)\}_{j=H,L}$, $MPK(z)$, $\{Q_j(z)\}_{j=H,L}$, $Q(z)$ and $\{\zeta_j(z)\}_{j=H,L}$. Using the above pricing functionals, we can combine the first order condition (37) and the envelope condition (39) as

$$E\left[\widetilde{M}(z,z')\{1+\zeta_H(z')+\zeta_L(z')\}\right]R_f(z;s') = E\left[\widetilde{M}(z,z')\{1+[\zeta_H(z)+\zeta_L(z)]\}Q(z)\right], \tag{50}$$

where $\widetilde{M}(z,z')$ is defined in Equation (38). To save notation, we do not write explicitly the information set based on which the expectations are taken. However, all expectations should be computed conditioning on the state variables $(z;s')$.

Assuming that $\mu^n(z')$ is the marginal value of net worth in the next period, the left-hand side of Equation (50) can now be written as:

$$\begin{aligned}
&\frac{E\left[\frac{\beta[m(z)-i]}{c^n(z')[1-\delta(z)+i(z)]}\{(1-\Lambda')+\Lambda'\mu^n(z')\}\{1+\zeta_H(z')+\zeta_L(z')\}\right]}{E\left[\frac{\beta[m(z)-i]}{c^n(z')[1-\delta(z)+i]}\right]} \\
=\;&\frac{E\left[\frac{\{(1-\Lambda')+\Lambda'\mu^n(z')\}}{c^n(z')}\{1+\zeta_H(z')+\zeta_L(z')\}\right]}{E\left[\frac{1}{c^n(z')}\right]}.
\end{aligned}$$

Similarly, the right-hand side of Equation (50) is

$$E\left[\frac{\beta[m(z)-i(z)]\{(1-\Lambda')+\Lambda'\mu^n(z')\}}{c^n(z')[1-\delta(z)+i(z)]}\{1+[\zeta_H(z')+\zeta_L(z')]\}Q(z')\right]$$

Therefore, Equation (50) can be written as:

$$\begin{aligned}
&E\left[\frac{\beta[m(z)-i(z)]\{(1-\Lambda')+\Lambda'\mu^n(z')\}}{c^n(z')[1-\delta(z)+i(z)]}\{1+[\zeta_H(z')+\zeta_L(z')]\}Q(z')\right] \\
=\;&\frac{E\left[\frac{\{(1-\Lambda')+\Lambda'\mu^n(z')\}}{c^n(z')}\{1+\zeta_H(z')+\zeta_L(z')\}\right]}{E\left[\frac{1}{c^n(z')}\right]},
\end{aligned} \tag{51}$$

or

$$\frac{m(z)-i(z)}{[1-\delta(z)+i(z)]}=\frac{E\left[\frac{\{(1-\Lambda')+\Lambda'\mu^n(z')\}}{c^n(z')}\{1+\zeta_H(z')+\zeta_L(z')\}\Big| z,s'\right]}{\beta E\left[\frac{\{(1-\Lambda')+\Lambda'\mu^n(z')\}}{c^n(z')}\{1+[\zeta_H(z')+\zeta_L(z')]\}Q(z')\Big| z,s'\right]\times E\left[\frac{1}{c^n(z')}\Big| z,s'\right]}. \tag{52}$$

Below we provide a summary of our recursive policy function iteration procedure.

Summary of the iteration procedure

1. Using Proposition 2 to construct the policy functions $\phi(z)$ and $u(z)$. This step is done independently of the iterations.

2. We start from an initial guess of the equilibrium functionals $\{c^0(z),\mu^0(z)\}$.

3. Given a set of equilibrium functionals, $\{c^n(z),\mu^n(z)\}$, we use equations (49) and (52) to jointly solve for $s'=s'(z)$ and $i(z)$.

4. Given $s'(z)\,i(z)$, we use the following equilibrium relationship to update $\{c^n(z),\mu^n(z)\}$:

$$\begin{aligned} c^{n+1}(z) &= m(z)-i(z), \\ \mu^{n+1}(z) &= E\left[\frac{\{(1-\Lambda')+\Lambda'\mu^n(z')\}}{c^n(z')}\{1+\zeta_H(z')+\zeta_L(z')\}\Big| z,s'(z)\right]/E\left[\frac{1}{c^n(z')}\Big| z,s'(z)\right]. \end{aligned}$$

5. Go back to step 3 and iterate until the error is smaller than a preset convergence criteria, ε, i.e., $\sup|c^{n+1}(z)-c^n(z)|+\sup|\mu^{n+1}(z)-\mu^n(z)|<\varepsilon$.