A Simple Proof of Theorem 2

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Here we provide a proof for Theorem 2 of the paper in a two-period model by assuming i) a finite state space, ii) fully revealing announcements, and iii) equal probability of each state. The proof in the paper shows that the conclusion of the theorem holds in a fully dynamic model without assuming fully revealing announcements. In addition, the assumption of a finite space with equal probability can be replaced by a continuum. Because the proof of Theorem 2 under the assumption of finite state space is relatively simple and does not require functional analysis in infinite dimensional spaces, we present such a proof in this note.

Under Assumptions i)-iii), the intertemporal preference can be written as $u(C_0) + \beta\mathcal{I}[u(C_1)]$, where the certainty equivalence functional $\mathcal{I}$ maps random variables into the real line. Because the probability space is finite, we can identify every random variable with a $N-$dimensional vector. We denote $V = [V_1, V_2, \cdots, V_N]$, where $V_s = u(C_{1,s})$ is the date-1 utility of the agent. We assume that the range of $u(C)$, denoted Ψ, is a closed interval on the real line. The set of all Ψ-valued random variables can be denoted as Ψ^N. As in the paper, we make the following assumptions on $\mathcal{I}$:

Assumption 1: $\mathcal{I}$ is continuously differentiable with strictly positive partial derivatives.
Assumption 2: $\mathcal{I}[k] = k$ whenever k is a constant.

As we show in equation (12) and (13) on page 9 of the paper,

$$P^- = E\left[m^*(s)\,P^+(s)\right], \tag{1}$$

where the A-SDF, $m^*(s)$ is given by:

$$m^*(s) = \frac{1}{\pi(s)} \frac{\frac{\partial}{\partial V_s}\mathcal{I}[V]}{\sum_{n=1}^{N} \frac{\partial}{\partial V_n}\mathcal{I}[V]}. \tag{2}$$

In the above equation, $\frac{\partial}{\partial V_s}\mathcal{I}[V]$ denotes the partial derivative of $\mathcal{I}[V]$ with respect to its *sth* element. Equation (1) implies that the announcement premium is positive (negative) if

$$E\left[m^*(s)\,P^+(s)\right] \leq (\geq)\, E\left[P^+(s)\right].$$

We first show that Condition 1 in the paper is equivalent to the "negative comonotonicity" of the partial derivatives of $\mathcal{I}[V]$:

Lemma 1 *The following two conditions are equivalent:*

1. *The announcement premium is non-negative for all payoffs that are comonotone with V.*[1]

2. *For any $V \in \Psi^N$,*
$$\left(\frac{\partial}{\partial V_s}\mathcal{I}[V] - \frac{\partial}{\partial V_{s'}}\mathcal{I}[V]\right)(V_s - V_{s'}) \le 0. \tag{3}$$

Proof. *First, we assume that 1) is true and prove 2) by contradiction. Suppose there exist V and s, s' such that $V_s > V_{s'}$ and $\frac{\partial}{\partial V_s}\mathcal{I}[V] > \frac{\partial}{\partial V_{s'}}\mathcal{I}[V]$. Consider the following payoff:*

$$X(n) = V_n \quad for \quad n = s, s'; \quad X(n) = 0 \quad otherwise.$$

Clearly, X is comonotone with V, and therefore positively correlated with $m^*(s)$ defined in (2). Therefore,
$$P^- = E[m^*(s) X(s)] > E[m^*(s)] E[X(s)] = E[X(s)],$$
contradicting a non-negative announcement premium.

Next, we assume that 2) is true and prove 1). Take any X that is comonotone with V, then

$$P^- = E[m^*(s) X(s)] \le E[m^*(s)] E[X(s)] = E[X(s)]$$

because $m^*(s)$ and $X(s)$ are negatively correlated. ■

Lemma 1 establishes the equivalence between non-negative announcement premium (for payoffs that are comonotone with continuation utility) and inequality (3). Inequality (3) is known to be a characterization of Schur concave functions, which is equivalent to monotone with respect to second order stochastic dominance for functions defined on finite probability spaces with equal probabilities. We summarize the equivalence results in the following lemma and refer the readers to Marshal and Okin or Muller and Stoyan for reference of such results.

Lemma 2 *For any $\mathcal{I}$ that satisfies Assumption 1, the following two statements are equivalent:*

1. *$\mathcal{I}[V]$ is non-decreasing in second order stochastic dominance if and only if for any $V \in \Psi^N$, $\left(\frac{\partial}{\partial V_s}\mathcal{I}[V] - \frac{\partial}{\partial V_{s'}}\mathcal{I}[V]\right)(V_s - V_{s'}) \le 0$.*

2. *$\mathcal{I}[V]$ is strictly increasing in second order stochastic dominance if and only if any $V \in \Psi^N$, $\left(\frac{\partial}{\partial V_s}\mathcal{I}[V] - \frac{\partial}{\partial V_{s'}}\mathcal{I}[V]\right)(V_s - V_{s'}) \le 0$, and strict inequality holds whenever $V_s \neq V_{s'}$.*

3. *$\mathcal{I}[V]$ is non-increasing in second order stochastic dominance if and only if any $V \in \Psi^N$, $\left(\frac{\partial}{\partial V_s}\mathcal{I}[V] - \frac{\partial}{\partial V_{s'}}\mathcal{I}[V]\right)(V_s - V_{s'}) \ge 0$.*

[1] Recall that a payoff X is comonotone with V if $\forall s$ and s' such that $X(s) \cdot X(s') \neq 0$, $[X(s) - X(s')][V(s) - V(s')] \ge 0$.

With the above we are ready to prove Theorem 2 in the paper. The first part of Theorem 2 is

1. *The announcement premium is zero for all assets if and only if I is expected utility.*

   Proof. If $\mathcal{I}$ is the expectation operator, that is, $\mathcal{I}[V] = \sum_{s=1}^{N} \pi(s) V(s)$, then by (2), $m^*(s) = 1$, and the announcement premium must be zero for all assets. Conversely, if the announcement premium is zero for all assets, we must have $m^*(s) = m^*(s')$ for all s, s', otherwise we can construct an asset with nonzero payoff in state s and s' that requires a non-trivial announcement premium. This implies that $\left(\frac{\partial}{\partial V_s}\mathcal{I}[V] - \frac{\partial}{\partial V_{s'}}\mathcal{I}[V]\right)(V_s - V_{s'}) = 0$ for all s,s'. For any $V \in \Psi^N$, note that $E[V] \geq_{SSD} V$, by the above lemma, we must have

   $$\mathcal{I}[V] = \mathcal{I}[E[V]] = E[V],$$

   where the last equality uses Assumption 2. ■

   The second part of Theorem 2 is a direct consequence of Lemma 1 and Lemma 2:

2. *The announcement premium is non-negative for all assets with payoffs comonotone with V if and only I is non-decreasing with respect to second order stochastic dominance.*

   From the above discussion, it is clear that a stronger version of the above result is also true, that is,

3. *The announcement premium is strictly positive for all assets with payoffs strongly comonotone with V if and only I is strictly increasing with respect to second order stochastic dominance.*